

***ALGEBRAS,
GROUPS
AND
GEOMETRIES***

FOUNDED IN 1984. SOME OF THE PAST EDITORS
INCLUDE PROFESSORS: A. A. ANDERSEN,
V. V. DEODHAR, J. R. FAULKNER,
M. GOLDBERG, N. IWAHORI, M. KOECHER,
M. MARCUS, K. M. MCCRIMMON, R. V. MOODY,
R. H. OHEMKE, J. M. OSBORN, J. PATERA,
H. RUND, T. A. SPRINGER, G. S. TSAGAS.

EDITORIAL BOARD

M.R. ADHIKARI
A.K. ARINGAZIN
P. K. BANERJI
A. BAYOUMI
QING-MING CHENG
A. HARMANCY
L. P. HORWITZ
C. P. JOHNSON
S.L. KALLA
N. KAMIYA
J. LOHMUS*
R. MIRON
M.R. MOLAEI
P. NOWOSAD
KAR-PING SHUM
SERGEI SILVESTROV
H. M. SRIVASTAVA
G.T. TSAGAS*

* In memoriam

**FOUNDER and
Editor In Chief
R. M. SANTILLI**



HADRONIC PRESS, INC.

ALGEBRAS GROUPS AND GEOMETRIES

VOLUME 32, NUMBER 4, DECEMBER 2015

MATRIX RINGS OVER GENERALIZED (σ, δ) -RIGID RINGS, 437

V.K. Bhat and Meeru Abrol

School of Mathematics, SMVD University

Katra, J and K, India-182320

MULTISCALE AND APPLIED SYSTEMS ANALYSIS OF A BRAIN ACTIVITY. I. GENERAL DESCRIPTION, 449

A. Makarenko

NTUU "KPI", Institute for of Applied System Analysis

Prospect Peremogy 37, 03056, Kiev-56, Ukraine

RETRACTNESS OF COMPACT REVERSIBLE DIRECTED COMPLETE POSET ACTS, 461

M. Mehdi Ebrahimi and Mahdieh Yavari

Department of Mathematics, Shahid Beheshti University, G.C.

Tehran 19839, Iran

ON THE PLANAR AND OUTER PLANAR ANNIHILATING-IDEAL GRAPHS OF A LATTICE, 479

F. Shamsavar

Department of Pure Mathematics, International Campus of Ferdowsi

University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

MATHEMATICAL MODELING OF SPATIAL AND TEMPORAL NONLOCALITIES ON NONLINEAR WAVES EVOLUTION, 495

Danylenko V.A., Danevich T.B., Makarenko O.S.,

Skurativsky S.I. and Vladimirov V.A.

Subbotin Institute of Geophysics of the National Academy of Sciences of

Ukraine, Palladin Av. 32, Kyiv, Ukraine, UA-03680

**PLANAR AND PROJECTIVE LINE GRAPHS ASSOCIATED
TO ZERO DIVISOR GRAPHS OF PARTIALLY ORDERED SETS, 525**

A. Parsapour

Department of Pure Mathematics, International Campus of Ferdowsi
University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

**MATHEMATICAL MODELING OF THE SYSTEM OF CHAINS OF
COUPLED ELEMENTS WITH STRONG ANTICIPATION, 545**

A. Makarenko

NTUU "KPI", Institute for of Applied System Analysis
Prospect Peremogy 37, 03056, Kiev-56, Ukraine

INDEX VOL. 32, 2015....571

MATRIX RINGS OVER GENERALIZED (σ, δ) -RIGID RINGS**V.K. Bhat and Meeru Abrol**

School of Mathematics, SMVD University

Katra, J and K, India-182320

vijaykumarbhat2000@yahoo.com

Received May 15, 2015

Abstract

Let R be a ring, σ an endomorphism of R and δ a σ -derivation of R . Recall that R is said to be a weak (σ, δ) -rigid ring if $a(\sigma(a) + \delta(a)) \in N(R) \Leftrightarrow a \in N(R)$ for $a \in R$ (where $N(R)$ is the set of nilpotent elements of R). In this paper we prove that a ring R is a weak (σ, δ) -rigid ring if and only if for any positive integer n , the $n \times n$ upper triangular matrix ring $T_n(R)$ is a weak (σ, δ) -rigid ring.

Let R be a ring, σ and δ as above. Then σ can be extended to an endomorphism (say $\bar{\sigma}$) of $T(R, R)$ and δ can be extended to a $\bar{\sigma}$ -derivation (say $\bar{\delta}$) of $T(R, R)$, where $T(R, R)$ is the trivial extension of R by R . (R is viewed as a bimodule over itself). We show that $T(R, R)$ is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring if and only if R is a weak (σ, δ) -rigid ring.

2010 Mathematics Subject Classification: 16W20, 16P40, 16S50

Key words: endomorphism, derivation, matrix ring, weak (σ, δ) -rigid ring

1 Introduction

All rings are associative with identity $1 \neq 0$, unless otherwise stated. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The set of natural numbers is denoted by $\mathbf{N}$, the ring of integers is denoted by $\mathbf{Z}$, the field of rational numbers is denoted by $\mathbf{Q}$, the field of real numbers is denoted by $\mathbf{R}$ and the field of complex numbers is denoted by $\mathbf{C}$, unless otherwise stated.

Now let R be a ring, σ an endomorphism of R . Recall that $\delta : R \rightarrow R$ an additive map such that

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R$$

is called a σ -derivation of R .

We recall the following definitions:

Definition 1: [6] Let R be a ring and M an R -bimodule. The trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

$T(R, M)$ is isomorphic to the the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$.

Definition 2:[4] An endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies that $a = 0$, for all $a \in R$. A ring R is said to be σ -rigid if there exists a rigid endomorphism σ of R .

Also σ -rigid rings are reduced rings by Hong et. al. [3]. The conjugacy map on $\mathbf{C}$ is a rigid automorphism.

Definition 3:[5] Let R be a ring and σ an endomorphism of R . Then R is said to be $\sigma(*)$ -ring if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$ for $a \in R$.

Example 1: (Example 1 of [5]) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field.

Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

Let $\sigma : R \rightarrow R$ be defined by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \text{ for all } a, b, c \in F.$$

Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$ -ring.

We note that the above ring is not σ -rigid. Let $0 \neq a \in F$. Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 2: Let $R = F[x]$ be the polynomial ring over the field F .

Let $\sigma : R \rightarrow R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$.

Then R is not a $\sigma(*)$ -ring.

Definition 4: [6] A ring R with an endomorphism σ is a weak σ -rigid ring if $a\sigma(a) \in N(R)$ implies and is implied by $a \in N(R)$ for $a \in R$.

Example 3: Let

$$R = \left\{ \begin{pmatrix} a+ib & c+id \\ 0 & e+if \end{pmatrix} \mid a, b, c, d, e, f \in \mathbf{R} \right\}.$$

Define an automorphism σ on R by

$$\sigma\left(\begin{pmatrix} a+ib & c+id \\ 0 & e+if \end{pmatrix}\right) = \begin{pmatrix} a-ib & c-id \\ 0 & e-if \end{pmatrix}.$$

Then R is a weak σ -rigid ring.

Example 4: Let F be a field and $R = F(x)$, the field of rational polynomials in one variable x over F . Then $N(R) = \{0\}$. Let $\sigma : R \rightarrow R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

Then R is not a weak σ -rigid ring. For let $f(x) = xa$, $f(0) = 0$ and $f(x)\sigma(f(x)) = xa.0 = 0 \in N(R)$. But $0 \neq f(x) \notin N(R)$.

Definition 5: [1] Let R be a ring. Let σ be an automorphism of R and δ a σ -derivation of R . Then R is a δ -ring if $a\delta(a) \in P(R)$ implies that $a \in P(R)$. Note that a δ -ring is without identity, $1 \neq 0$ as $1.\delta(1) = 0$, but $1 \neq 0$.

Definition 6:[2] Let R be a ring. Let σ be an automorphism of R and δ a σ -derivation of R . Then R is a δ -rigid ring if $a\delta(a) = 0$ implies that $a = 0$ for $a \in R$. Note that a ring R with identity is not a δ -rigid ring as $1.\delta(1) = 0$, but $1 \neq 0$.

Example 5: Let $R = \mathbf{R} - \{1, -1\}$ be a ring without identity. Here $P(R) = \{0\}$.

$\sigma : R \rightarrow R$ is an automorphism defined by

$$\sigma(a) = a^{-1}, \text{ for all } a \in R.$$

Define $\delta : R \rightarrow R$ by

$$\delta(a) = a - \sigma(a) \text{ for } a \in R.$$

Then δ is a σ -derivation of R . Let $a \neq 0 \in R$ be such that $a\delta(a) \in P(R)$. Also $a(a - a^{-1}) \in P(R) = \{0\}$ or $a^2 - 1 = 0$ which implies that $a = 1, -1$. But this is a contradiction. Hence $a = 0$. Thus R is a δ -ring.

Example 6: Let $R = (a_{ij})_{2 \times 2}$, the set of all 2×2 matrices over the ring $n\mathbf{Z}$, $n > 1$, with $a_{21} = 0$. Then $\sigma : R \rightarrow R$ is an automorphism defined by $\sigma((a_{ij})) = (b_{ij})$, where $b_{ij} = a_{ij}$ except that $b_{12} = -a_{12}$. Define $\delta : R \rightarrow R$ by $\delta((a_{ij})) = (c_{ij})$, where $c_{ij} = 0$ except that $c_{12} = 2a_{12} + a_{22} - a_{11}$. Then it can be seen that δ is a σ -derivation of R . But R is not a δ -rigid ring, as for $A = (a_{ij})_{2 \times 2}$ with $a_{ij} = 0$, except $a_{22} = 1$, $A\delta(A) = (0)$.

2 Preliminaries

Definition 7: Let R be a ring. Let σ be an endomorphism of R and δ a σ -derivation of R . Then R is said to be a weak (σ, δ) -rigid ring if $a(\sigma(a) + \delta(a)) \in N(R) \Leftrightarrow a \in N(R)$ for $a \in R$.

Example 7: Let $R = \mathbf{Z}[\sqrt{2}]$. Then $\sigma : R \rightarrow R$ defined as

$$\sigma(a + b\sqrt{2}) = a - b\sqrt{2} \text{ for } a + b\sqrt{2} \in R$$

is an endomorphism of R . For any $s \in R$. Define $\delta_s : R \rightarrow R$ by

$$\delta_s(a + b\sqrt{2}) = (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2}) \text{ for } a + b\sqrt{2} \in R.$$

Then δ_s is a σ -derivation of R . Here $N(R) = \{0\}$.

Let $(a + b\sqrt{2})\{\sigma(a + b\sqrt{2}) + \delta_s(a + b\sqrt{2})\} \in N(R)$

which gives $(a + b\sqrt{2})\{(a - b\sqrt{2}) + (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2})\} \in N(R)$

or $(a + b\sqrt{2})\{a - b\sqrt{2} + as + bs\sqrt{2} - sa + sb\sqrt{2}\} \in N(R)$.

Hence $(a + b\sqrt{2})\{a + (2s - 1)b\sqrt{2}\} \in N(R) = \{0\}$ which gives $a = 0, b = 0$ or $a + b\sqrt{2} = 0 + 0\sqrt{2} \in N(R)$. Thus R is a weak (σ, δ) -rigid ring.

Remark 1: Let σ be an endomorphism of R and δ a σ -derivation of R . Then σ can be extended to an endomorphism (say $\bar{\sigma}$) of $T_n(R)$ by

$$\bar{\sigma}\left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}\right) = \begin{pmatrix} \sigma(a_{11}) & \sigma(a_{12}) & \dots & \sigma(a_{1n}) \\ 0 & \sigma(a_{22}) & \dots & \sigma(a_{2n}) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma(a_{nn}) \end{pmatrix},$$

$$\text{for any } \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \in T_n(R).$$

Also δ can be extended to a $\bar{\sigma}$ -derivation (say $\bar{\delta}$) of $T_n(R)$ by

$$\bar{\delta}\left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}\right) = \begin{pmatrix} \delta(a_{11}) & \delta(a_{12}) & \dots & \delta(a_{1n}) \\ 0 & \delta(a_{22}) & \dots & \delta(a_{2n}) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \delta(a_{nn}) \end{pmatrix},$$

$$\text{for any } \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \in T_n(R).$$

With this we prove the following :

Theorem A: Let σ be an endomorphism of a ring R and δ a σ -derivation of R . Then the following statements are equivalent:

1. R is a weak (σ, δ) -rigid ring.
2. $T_n(R)$ is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring.

(This has been proved in Theorem (3)).

Remark 2: Let σ be an endomorphism of ring R . Then σ can be extended to an endomorphism $\bar{\sigma} : T(R, R) \rightarrow T(R, R)$ defined by

$$\bar{\sigma}\left(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \sigma(r) & \sigma(m) \\ 0 & \sigma(r) \end{pmatrix}, \text{ for any } \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in T(R, R).$$

Let δ be a σ -derivation of R . Then it can be extended to a $\bar{\sigma}$ -derivation (say $\bar{\delta}$) by

$$\bar{\delta}\left(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \delta(r) & \delta(m) \\ 0 & \delta(r) \end{pmatrix}.$$

Here R is viewed as a bimodule over itself. There should be no confusion with $\bar{\sigma}$ and $\bar{\delta}$ as in Remark (2).

With this we prove the following:

Theorem B: Let σ be an endomorphism of ring R and δ a σ -derivation of R . Then the trivial extension $T(R, R)$ of R by R is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring if and only if R is a weak (σ, δ) -rigid ring. (R is viewed as a bimodule over itself). (This has been proved in Theorem (3)).

3 Proof of Main Results

Theorem 1: Let σ be an endomorphism of a ring R and δ a σ -derivation of R . Then the following statements are equivalent:

1. R is a weak (σ, δ) -rigid ring.
2. $T_n(R)$ is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring.

Proof: (1) $\Rightarrow$ (2) Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \in T_n(R)$

be such that $A \in N(T_n(R))$. Then there exists some positive integer n such that

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}^n = \begin{pmatrix} a_{11}^n & * & \dots & * \\ 0 & a_{22}^n & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}^n \end{pmatrix} = 0,$$

where $*$ are non-zero terms involving summation of powers of some or all of $a_{ij}, i, j \in \mathbf{Z}, 1 \leq i, j \leq n$. Thus $a_{ii} \in N(R), i = 1, 2, \dots, n$. Since R is a weak (σ, δ) -rigid ring, we have $a_{ii}(\sigma(a_{ii}) + \delta(a_{ii})) \in N(R)$. So there exists some positive integer t_i such that $(a_{ii}(\sigma(a_{ii}) + \delta(a_{ii})))^{t_i} = 0, i = 1, 2, \dots, n$. Let $t = \max\{t_i\}, i = 1, 2, \dots, n$. Then

$$\begin{aligned} & \left[\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \left\{ \overline{\sigma} \left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right) + \right. \\ & \left. \overline{\delta} \left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right) \right\} \right]^{tn} \\ &= \begin{pmatrix} a_{11}(\sigma(a_{11}) + \delta(a_{11})) & * & \dots & * \\ 0 & a_{22}(\sigma(a_{22}) + \delta(a_{22})) & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}(\sigma(a_{nn}) + \delta(a_{nn})) \end{pmatrix}^{tn} \end{aligned}$$

$$= \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}^n = 0, \text{ where } * \text{ are non-zero terms involving sum-}$$

mation of powers of some or all of $a_{ij}, \sigma(a_{ij}), \delta(a_{ij}), i, j \in \mathbf{Z}, 1 \leq i, j \leq n$.
Hence

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \left\{ \bar{\sigma} \left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right) + \right. \\ \left. \bar{\delta} \left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right) \right\} \in N(T_n(R)).$$

Now let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \in T_n(R)$$

be such that $A[\bar{\sigma}(A) + \bar{\delta}(A)] \in N(T_n(R))$. Then there is some positive integer n , such that

$$\left[\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \left\{ \bar{\sigma} \left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right) + \right. \right. \\ \left. \left. \bar{\delta} \left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right) \right\} \right]^n$$

$$= \begin{pmatrix} a_{11}(\sigma(a_{11}) + \delta(a_{11})) & * & \dots & * \\ 0 & a_{22}(\sigma(a_{22}) + \delta(a_{22})) & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}(\sigma(a_{nn}) + \delta(a_{nn})) \end{pmatrix}^n$$

$= 0$, where $*$ are non-zero terms involving summation of powers of some or all of $a_{ij}, \sigma(a_{ij}), \delta(a_{ij}), i, j \in \mathbf{Z}, 1 \leq i, j \leq n$, which implies that $a_{ii}(\sigma(a_{ii}) + \delta(a_{ii})) \in N(R), i = 1, 2, 3, \dots, n$. Since R is a weak (σ, δ) -rigid ring, this gives $a_{ii} \in N(R), i = 1, 2, 3, \dots, n$ and so there is some positive integer t_i such that $(a_{ii})^{t_i} = 0$. Let $t = \text{Max}\{t_i\}, i = 1, 2, 3, \dots, n$, then

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}^{tn} = \begin{pmatrix} a_{11}^t & * & \dots & * \\ 0 & a_{22}^t & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}^t \end{pmatrix}^n = \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}^n$$

$= 0$, where $*$ are non-zero terms involving summation of powers of some or all of a_{ij} . Hence $A \in N(T_n(R))$. Therefore, $T_n(R)$ is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring.

(2) $\Rightarrow$ (1) Let $a \in N(R)$, then $a^n = 0$, for some positive integer n . Let

$$A = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in T_n(R), \text{ then } A \in N(T_n(R)). \text{ Since } T_n(R)$$

is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring,

$$\begin{aligned} & \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \left[\bar{\sigma} \left(\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) + \bar{\delta} \left(\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) \right] \\ &= \begin{pmatrix} a(\sigma(a) + \delta(a)) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in N(T_n(R)). \end{aligned}$$

Thus $a(\sigma(a) + \delta(a)) \in N(R)$. Now let $a(\sigma(a) + \delta(a)) \in N(R)$, then

$$\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \left[\bar{\sigma} \left(\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) + \bar{\delta} \left(\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) \right] \\ \in N(T_n(R)). \text{ Since } N(T_n(R)) \text{ is a weak } (\bar{\sigma}, \bar{\delta})\text{-rigid ring, we have}$$

$$\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in T_n(R) \text{ and so } a \in N(R).$$

Hence R is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring.

Theorem 2: Let σ be an endomorphism of ring R and δ a σ -derivation of R . Then the trivial extension $T(R, R)$ of R by R is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring if and only if R is a weak (σ, δ) -rigid ring (R is viewed as a bimodule over itself).

Proof: Since $T(R, R)$ is isomorphic to the subring $\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r, m \in R \right\}$ of a ring $T_2(R)$ as in Theorem (3). Also each subring of a weak (σ, δ) -rigid ring is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring. So it is easy to see that $T(R, R)$ is a weak $(\bar{\sigma}, \bar{\delta})$ -rigid ring if and only if R is a weak (σ, δ) -rigid ring.

References

- [1] V. K. Bhat, On 2-primal Ore extensions, Ukr. Math. Bull., Vol. 4(2)(2007), 173-179.
- [2] Y. Hirano, On the uniqueness of rings of coefficients in skew polynomial rings, Publ. Math. Debrecen, Vol. 54(3, 4)(1999), 489-495.
- [3] C.Y.Hong, N.K.Kim and T.K.Kwak, Ore extensions of Baer and p.p. - rings, J. Pure and Appl. Algebra, Vol. 151(3)(2000), 215-226.

- [4] J. Krempa, Some examples of reduced rings, Algebra Colloq., Vol. 3(4)(1996), 289-300.
- [5] T.K.Kwak, Prime radicals of Skew polynomial rings, Int. J. Math. Sci., Vol. 2(2)(2003), 219-227.
- [6] L. Ouyang, Extensions of generalized α -rigid rings, Int. Electron. J. Algebra, Vol. 3(2008), 103-116 .

**MULTISCALE AND APPLIED SYSTEMS ANALYSIS OF A
BRAIN ACTIVITY. I. GENERAL DESCRIPTION¹****A. Makarenko**

NTUU “KPI”, Institute for of Applied System Analysis
Prospect Peremogy 37, 03056, Kiev-56, Ukraine
makalex@i.com.ua

Received February 10, 2015

Abstract

Symbiosis of theoretical physics, biology and information technologies got to all regions of science of life. However among all new ideas, conceptions, applications, are most essential related to the processes what is going on in the brain of man and, accordingly, to different activity development in to the brain, and also the illnesses caused by violation of cerebral activity. List of the illnesses related to cerebral activity vast enough. So in proposed talk the analysis of architecture, dynamics and desirable structure of models is conducted for processes at different levels of hierarchy into the brain. The review of hierarchy patterns and structures of brain activities are proposed. The prospects of the use of new methods of prediction of attacks and recognition of seizures are described, objects based on construction in the extended phase plane using senior derivative. The prospects of the use of models of cellular automats are analyzed for processes in to the brain and the ground of their application is offered for the concrete phenomena. Some aspects of consciousness models are considered. The first part is devoted to the general system aspects of problems.

¹Based on invited talks given by A. Makarenko at the 6th Petrov International Symposium on *High Energy Physics, Cosmology and Gravity*, Kyiv–Kosivska Poliana (Ukraine), September 5–20, 2013 and at INDS11 (Klagenfurt, Austria).

1 Introduction

1.1 Common description of actuality

As far as development of society modern scientific methodology gets to all segments of public life. One of such segments is medicine, which more is based on the scientific pictures of man, device of human body and knowledge about processes in him, in connection with the newest achievements of computers sciences and information technologies. Symbiosis of medical science and information technologies got to all regions of practical medicine. However among all new ideas, conceptions, applications, are most essential related to the processes what is going on in the brain of man and, accordingly, to illnesses developing in to the brain, and also the illnesses caused by violation of cerebral activity. The necessity of such applications is conditioned above all things by that the processes of cerebral activity are, perhaps, one of most difficult in all human body. List of the illnesses related to cerebral activity vast enough: illness of Alzheimer, illness of Parkinson, epilepsy, psychical disorders and many others. Part of these illnesses has expressly visible indicators in the structure of changes of fabrics of brain (for example, at illness of Alzheimer), part (for example, psychical illnesses) by virtue of the complication yet only on the initial stages of understanding their neuro-correlates in activity, and the state of businesses in the study of some other already just is on a stage, ripening for application of methods, which fine recommended itself in physics, applied mathematics and management. Among such illnesses, which ripened for more wide study by the methods of physics, systems analysis, mathematical design data and signals processing, we will mark epilepsy and illness of Parkinson. Therefore in the real report the results of research of these illnesses are resulted from point of systems analysis and possibilities of design of processes at these illnesses, and also some possibilities of the practical use.

1.2 Description of illness of Parkinson and epilepsy and accessible information about them (different aspects)

In spite of relatively wide prevalence of epilepsy (to 1% all population) and illness of Parkinson (for example, by estimations of a to 3% adult population of the USA), the generally accepted determinations (and the more so formal) of these illnesses are not present, and at application of systems analysis to be based on diffuse enough definitions which are quintessence of experience of vast medical practice and enough large array of experimental neurophysiological and physical researches. It is possible to forecast, that the more exact (possibly even quantitative determinations even for the separate types of illnesses) will appear as a result of penetration of methods of exact sciences (physics, mathematics, information technologies, etc.). However for the researchers of such illnesses elementary their description is frequently needed from point of standard medicine. Thus by virtue of many-sided nature of the phenomena often anymore diffuse description befits from widely accessible sources. We will mark Wikipedia, Scholapedia, sites of the specialized medical associations, institutes, conferences, scientific pages of general sites and other above all things [1-9]. So, for epilepsy we will bring for information such sources over [4]: "Epilepsy (on Greece: grasped, caught, overtaken; at latin: 'epilepsia' or 'caduca') is one of the most widespread chronic neurological diseases of man, showing up in predisposition of organism to the sudden origin of convulsive attacks." (Wikipedia - Epilepsy). From Scolapedia [8]: "Epilepsy is a chronic neurological disorder characterized by occurrence and recurrence of seizures; epilepsy is thus and seizure of disorder. A seizure in turn is and transient of signs and/or symptoms due to abnormal excessive or synchronous neuronal activity in the brains [9]. Epilepsy is the third most common neurological disorder affecting more than 50 million people worldwide (<http://www.who.int/mediacentre/factsheets/fs999/en/index.html>). The seizures of about 30 to 40 percent of epilepsy patients do not respond to drugs, and percentage that has remained relatively stable despite significant efforts to develop new antiepileptic medication over the past decade (<http://www.epilepsyfoundation.org/about/factsfigures.cfm>). A recent and particularly powerful approach towards the understanding of seizure dynam-

ics is the combination of electrophysiology with high-resolution fluorescent imaging. Seizure mechanisms are too complex to understand without incorporating such measurements and observations into computational models. Theoretical predictions are quantitative in nature and allow direct comparison with experiments as access to various variables and parameters becomes increasingly available. Computational models are the most suitable tools to tie the advances made at various levels in epilepsy. Indeed, epileptic seizure is an example of a phenomenon which we feel cannot be properly understood without the lens that computational theory provides [10].” From Wikipedia [11]: “Parkinsons disease is the chronic disease characteristic for the persons of senior age group [12]. Caused by making progress destruction and death of neurons of black matter of middle brain and other departments of the central nervous system, using as dophamin as the neuro-mediator [13]. For illness of Parkinson motive violations are characteristic: tremor, hypokinezy, muscle rigidity, postural instability, and also vegetative and psychical disorders [14], is the result of decline of the braking influencing pale ball (palladium) located in the front department of cerebrum, on the striped body (striatum)”. And, one of the best [15]: “What is it? Parkinsons disease is a brain disorder that affects the motor system, the system that initiates and controls skeletal muscle movements. Parkinsons is chronic and progressive, meaning that it gets worse over time. The disease progresses at different rates in different people and affects each person uniquely some people may suffer incapacitating movement problems while others may experience only minor impairments for some time. It is impossible to predict which symptoms will affect any one person, but the primary symptoms of Parkinsons are:

- Tremor, trembling in the hands, arms, legs, jaw and face;
- Rigidity, stiffness of the limbs and trunk;
- Bradykinesia, slowness of movement;
- Akinesia, difficulty in initiating movement;
- Postural instability, impaired balance and coordination.

A number of other symptoms may also accompany Parkinsons disease, some minor and some that can be debilitating. These may include depression, changes in emotions or cognitive (thinking) abilities, difficulty in swallowing and chewing, speech changes, urinary problems or constipation, skin problems such as very oily or very dry skin, or excessive sweating, and sleep problems. Not all these symptoms occur in everyone with Parkinsons disease. Parkinsons disease is classified as a neurodegenerative disease, which means that it is marked by the progressive degeneration and death of certain groups of nerve cells, or neurons. (Other neurodegenerative disorders include Alzheimers disease, Huntingtons disease, and amyotrophic lateral sclerosis, or Lou Gehrigs disease.) In the case of Parkinsons, the neurons that die produce dopamine, an important brain chemical that is critical to the initiation and control of skeletal muscle movements. What Causes Parkinsons? While scientists dont yet know what triggers Parkinsons disease, they do know that the symptoms are the result of the progressive death or impairment of and discrete group of nerve cells in an area of the brain called the substantia nigra. Normally, these neurons produce dopamine, and neurotransmitter that acts as a chemical messenger to relay signals between the substantia nigra and another part of the brain called the striatum, which is part of the basal ganglia. As the dopamine neurons die, the level of dopamine decreases, which causes nerve cells to fire abnormally and excessively. This disrupts the brains ability to initiate and control muscle movements, causing the classic symptoms of Parkinsons disease.” [15], p.6-7. Vast clinical, experimental, including neurophysiological, and research base on illnesses, models and other in vivo experiments, and also on the design distributed on the enormous amount of sources. Partly their review is resulted in next sections.

1.3 Questions of systems analysis and decision of problem of illness of Parkinson and epilepsy

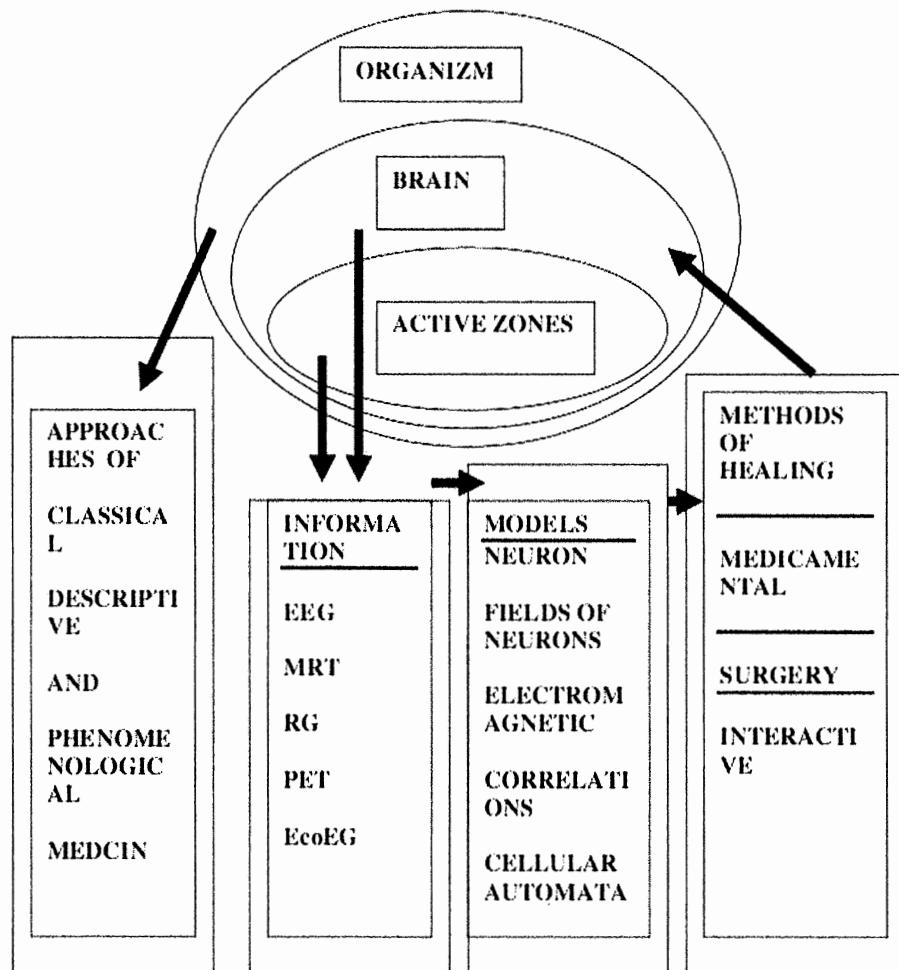
Coming all present knowledge from it is visible about the examined illnesses, that such phenomena are very difficult, maniaspects, engaging simultaneously in co-operation a lot of different subsystems of organism, thus both on different spatial and temporal levels scales. Such variety of the phenomena

requires both the use of integrative (including integral, global approaches) and necessity of understanding of separate subsystems, for what except for the use of direct researches all large meaningfulness acquires the use of mathematical design, thus frequently with bringing in of mathematical models quite different on a structure. We will notice that here by virtue of large complication of the systems and processes it is necessary to adhere to principle designing local tasks, it is necessary to think globally, taking into account on possibility all present information (theoretical and experimental). On Picture.1 the illustration, related to intercommunications of different facilities methods of research and treatment of illnesses of brain, is resulted. Accordingly, for development and raising of tasks of research of the future practically working algorithms used for treatment, the analysis of the following components related to each of blocks is needed: taken off information, possible architecture of subsystems, processes and methods of their design [16]. The next sections of this paper are devoted to consideration of such components, thus considerable support is done on the analysis of applicability of the most perspective (including offered by the author with colleagues) methods.

2 Applied Systems Analysis of Processes in a Brain

2.1 Some pictures of system organization of architecture of a brain

At the study of brain as systems very importantly knowledge about architecture of object, i.e. of what parts he consists and how these parts are linked between itself. Thus multi-scales and hierarchicalness of structure of brain is a substantial factor. The first level of architecture and geometry is usually well visible in classic medical researches – i.e. visible separate regions of brain: bark, hemispheres, system of blood supply and many other [1, 2]. At following hierarchical level one of basic systems units is selected: neuron, fields of neurons, nerves from other organs of body, gliya and other [3-5]. Usually objects exactly this level: neurons, to their connection, activ-



ity of the fields of neurons, and also loud speaker of separate elements and subsystems, make considerable part of modern researches tasks. However on possibility it is necessary to take into account that a few levels of hierarchy actually are, many of which are partly described in literature [3-7]. So, from the beginning confessed that a neuron is a difficult object, with in a number of chemical reactions, both in membranes and in insides. However much careful experimental investigations (by microscopy and other) set the difficult hierarchical underlying structure of separate neuron. Most interesting consists in that into a neuron there is difficult texture from micro-tubules, and each such micro-tubes is constructed as cylinder the surface of which consists of yet less tubes. And, further, each of such less micro-tubes is the linear chain let of cells, each of which can be in two states (C. Hammeroff, R. Penrose). from one electrode for example at research of one neuron or information in real time at direct surgical operations on a brain. There are also separate data, indirectly related to the processes in a brain in cognitive sciences, psychology, physiology and sciences, about the conduct, arrays. Presently during experimental researches the numerous cards of activity areas of cerebrum, proper to the different terms, are got, to influences and the decided man of tasks. As it applies to the tasks of fight against illnesses there is also a lot of specific information which is very important exactly for concrete illnesses. Separately it should be noted the newest researches (experimental and theoretical) on application of graph-theoretical approach to the processes in a brain, when the structures of brain appear as counts; descriptions of such counts are measured and dependence of processes is explored on the values of such descriptions (indexes) [8-10]. We will notice that so-called small is the most often used structure world model [8-10].

2.2 Basic studied processes in neurophysiology of a brain

In accordance with the picture of researchers of structure of brain, strategies of study of dynamic processes in a brain was built. It is necessary to notice that the studied tasks largely depended on an accessible experimental apparatus algorithms of treatment, and also from the pictures of physical essence of processes and, accordingly from their models. All foregoing depends

on the measured sizes terms of their measuring. As well-known methods we will mention EEG, fMRT, EcoG, PET, MEG, EEG (electroencephalogram), fMRT (functional magnetic resonance tomography), EcoG (electro - corticography), PET (positron emission tomography), MEG (magnetoencephalogram)) and other. As a result of measuring the large volume of information (as a rule, from many channels) which after is already processed by the chosen algorithms accumulates on these methods, or serves as initial material for construction or tuning of models. Interesting information supply with the inter-cranial measuring from one electrode for example at research of one neuron or information in real time at direct surgical operations on a brain. There are also separate data, indirectly related to the processes in a brain in cognitive sciences, psychology, physiology and sciences, about the conduct, arrays. Presently during experimental researches the numerous cards of activity areas of cerebrum, proper to the different terms, are got, to influences and the decided man of tasks. As it applies to the tasks of fight against illnesses there is also a lot of specific information which is very important exactly for concrete illnesses. Separately it should be noted the newest researches (experimental and theoretical) on application of graph-theoretical approach to the processes in a brain, when the structures of brain appear as counts; descriptions of such counts are measured and dependence of processes is explored on the values of such descriptions (indexes) [8-10]. We will notice that so-called small is the most often used structure world model [8-10].

2.3 Features of processes at epilepsy and illness of Parkinson, requiring an adequate account at the design and use of information technologies

The analysis of literature [11-14] showed that the presence of plenty of multi-frequency excitations is the first feature of the taken information off from one side structured (for example, spikes are conducts of decisions as he sharp troop landing), and, from other side, always present high-frequency oscillations of small amplitude). We will notice that experience of successful application of prognosis methods requires in one or another degree of adequate account of high-frequency vibrations (information about which

can get lost at application of methods of long-term averaging - for example, at the reconstruction of attractors or calculation of middle correlation dimension). In addition, as indicated, for example in [11, 12], the change of statistical conformities to the law of objects during the record of signals also results in a necessity to watch the change on small time domains. The second specific of features behaves to the different types of synchronization exposed algorithmically and by sight. So, alike, the experiments [13, 14] show the origin of synchronization in the different moments of development of epileptic attacks. Moreover, general epilepsy possibly is the extreme case of synchronization (or vice versa chaotic desynchronization). For construction of models the additional supervisions (for example [15]) are important, where the conclusion is done about non-diffusive character of distribution of brain activity and about some specific ways (channels) of conductivity of excitations, including between the fields of neurons in a brain. On the departments of cerebrum distribution of activity is another recently set feature of distribution of activity as avalanche growth of excitations at some processes [16]. Such conduct is acknowledged different from synchronization, waves of activity, vibrations and can act important part in functioning of brain. All these features of information set from point of experiment require further researches; understandings both the physical phenomenon and from point of treatment of information, functioning and decision-making. The decision of such tasks is frequently impossible without the use of the proper models. Therefore in a next chapters we bring the critical analysis over some from the existent classes of models from point of possibility of explanation of existent experimental information and will describe some models and researches tasks following from such analysis.

References

- [1] Penfield, W.; Jaspers, H. Epilepsy and the functional anatomy of the human brain; Churchill: London, 1954.
- [2] www.epilepsyfoundation.org
- [3] WWW Parkinson Association site

- [4] 1. Wikipedia Paper Epilepsi
- [5] Commission on Epidemiology and Prognosis, International League Against Epilepsy (1993). Guidelines for epidemiologic studies on epilepsy. Commission on Epidemiology and Prognosis, International League Against Epilepsy. *Epilepsia* 34 (4): 5926. DOI:10.1111/j.1528-1157.1993.tb00433.x. PMID 8330566.
- [6] Blume W, L?ders H, Mizrahi E, Tassinari C, van Emde Boas W, Engel J (2001). Glossary of descriptive terminology for ictal semiology: report of the ILAE task force on classification and terminology. *Epilepsia* 42 (9): 12128. DOI:10.1046/j.1528-1157.2001.22001.x. PMID 11580774.
- [7] 1. Viktor M., Popper A. Handbook on neurology 7th edition. M. Medical Inform agency., 2006. 677 . ISBN 5-89481-275-5 (translation from English)
- [8] 1. Models of epilepsy Steven J. Schiff and Ghanim Ullah (2009), Scholarpedia, 4(7):1409. doi:10.4249/scholarpedia.1409
- [9] Fisher, R. S., Van Emde Boas, W., Blume, W., Elger, C., Genton, P., Lee, P., and Engel, J. Jr. (2005) Epileptic Seizures and Epilepsy: Definitions Proposed by the International League Against Epilepsy (ILAE) and the International Bureau for Epilepsy (IBE). *Epilepsia*.
- [10] Mitra, P. and Bokil, H. (2007) Observed brain dynamics. Oxford University Press, USA.
- [11] WIKIPEDIA Paper: Parkinson diseases
- [12] Samii A, Nutt JG, Ransom BR. Parkinson's disease // *Lancet*. 2004. . 363. . 17831793. PMID 15172778.
- [13] Yachno N.N., Stulman D.R.. Neural systems diseases. M. Medicina. - 2001. Vol.. 2. pp.. 76-95. 744 p. (in Russian).
- [14] 1. Gusev E.I., Konovalov A.N. at al.. Neurology. M. Media. 2009. 2116 p.

- [15] What it Parkinson's disease? Brochure of The Michael Stern Parkinsons Research Foundation. Program in Molecular and Cellular Neuroscience at the Rockefeller University, N.Y., USA, 2012. 22. p.
- [16] Soltesz, I., and Staley, K. (2008) Computational neuroscience in epilepsy. Academic Press, UK.

**RETRACTNESS OF COMPACT REVERSIBLE
DIRECTED COMPLETE POSET ACTS****M. Mehdi Ebrahimi and Mahdiah Yavari**Department of Mathematics
Shahid Beheshti University, G.C.
Tehran 19839, Iran

Received September 2, 2015

Abstract

Actions of a monoid or a group on a set have always been of interest to mathematicians and computer scientists. On the other hand, domain theory, which studies directed complete partially ordered sets, was introduced by Scott in the 1970s as a foundation for programming semantics and provides an abstract model of computation, and has grown into a respected field on the borderline between mathematics and computer science.

In this paper, combining the above two notions, we consider actions of a semigroup (monoid or group) on compact reversible directed complete posets and study the algebraic notion of retractness, which is tightly connected to the study of injectivity, with respect to monomorphisms and embeddings in the category so obtained.

Keywords: Action of a monoid, directed complete poset, absolute retract, injective object.

1 Introduction

Recall that for a class $\mathcal{M}$ of monomorphisms in a category $\mathcal{C}$, an object $A \in \mathcal{C}$ is called $\mathcal{M}$ -*injective* if for each $\mathcal{M}$ -*morphism* $f : B \rightarrow C$ and morphism $g : B \rightarrow A$ there exists a morphism $h : C \rightarrow A$ such that $hf = g$.

Also, an object A of $\mathcal{C}$ is called $\mathcal{M}$ -*absolute retract* if it is a retract of each of its $\mathcal{M}$ -extensions; that is, for each $\mathcal{M}$ -morphism $f : A \rightarrow C$ there exists a morphism $h : C \rightarrow A$ such that $hf = id_A$, in which case h is said to be an $\mathcal{M}$ -*retraction*.

We simply call $\mathcal{M}$ -injective ($\mathcal{M}$ -absolute retract) objects as *injective* (*absolute retract*), if $\mathcal{M}$ is the class of all monomorphisms. Clearly, injective objects are absolute retracts. One would like to have “ $\mathcal{M}$ -injectivity to be equivalent to $\mathcal{M}$ -absolute retractness (see [3, 5, 7, 8, 10, 11, 12, 14]), which is not always the case.

In this paper, we consider $\mathcal{M}$ to be the class of all monomorphisms or the class $\mathcal{E}$ of all embeddings in the categories **C-R-Dcpo-S** of compact reversible directed complete poset acts and **C-SR-Dcpo-S** of compact strongly reversible directed complete poset acts, which will be defined below. Then, we will show that, contrary to the existence of non-trivial absolute retracts, there does not exist any non-trivial injective object in these categories.

2 Preliminaries

First we recall some preliminaries needed in the sequel. The reader can find more details in [1, 2, 4, 6, 9, 13, 15, 16]. Let **Pos** denote the category of all partially ordered sets (posets) with order-preserving (monotone) maps between them. A non-empty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$; and P is called *directed complete*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P . A subset X of a dcpo P is called a *sub dcpo* of P if for each $D \subseteq^d X$, $\bigvee^d D$ exists in X .

A *dcpo* or a *continuous map* $f : P \rightarrow Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. Thus we have the category **Dcpo** of all dcpo's with continuous maps between them.

A *po-monoid* (*po-semigroup*, *po-group*) S is one with a partial order $\leq$ which is compatible with its binary operation (that is, for $s, t, s', t' \in S$, $s \leq t$ and $s' \leq t'$ imply $ss' \leq tt'$). Similarly, a *dcpo-monoid* (*group*) is one which is also a dcpo whose binary operation is continuous.

Recall that a (*right*) S -act or S -set for a monoid S is a set A equipped with an action $A \times S \rightarrow A$, $(a, s) \rightsquigarrow as$, such that $ae = a$ (e is the identity element of S) and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act**- S denote the category of all S -acts with the action preserving maps ($f : A \rightarrow B$ with $f(as) = f(a)s$, for all $a \in A, s \in S$). Let A be an S -act. An element $a \in A$ is called a *zero*, *fixed*, or a *trap element* if $as = a$, for all $s \in S$.

For a po-monoid S , a (*right*) S -poset is a poset A which is also an S -act whose action $\lambda : A \times S \rightarrow A$ is order-preserving, where $A \times S$ is with componentwise order. The category of all S -posets with action preserving monotone maps between them is denoted by **Pos**- S .

Also, for a dcpo-monoid S , a (*right*) S -dcpo is a dcpo A which is also an S -act whose action $\lambda : A \times S \rightarrow A$ is continuous.

A non-empty subset B of an S -dcpo A is called a *sub S -dcpo* of A if B is a sub dcpo and subact of A . In this case, A is said to be an *extension* of B .

By an *S -dcpo map* between S -dcpo's, we mean a map $f : A \rightarrow B$ which is both continuous and action preserving. We denote the category of all S -dcpo's and S -dcpo maps between them by **Dcpo**- S .

A morphism $f : A \rightarrow B$ in the category of all S -dcpo's is called *order-embedding*, or briefly *embedding*, if for all $x, y \in A$, $f(x) \leq f(y)$ if and only if $x \leq y$.

Because of the fact that sub S -dcpo's are exactly subsets for which the inclusion map is order-embedding, we consider an arbitrary order-embedding as an inclusion from a sub S -dcpo.

3 Injectivity and retractness of compact reversible dcpo acts

In this section we study injectivity and absolute retractness in the category of compact reversible directed complete poset acts and we show that injectivity is not the same as absolute retractness. Also, we characterize $\mathcal{E}$ -injective and $\mathcal{E}$ -absolute retract objects. First, we define some notions needed in this sequel.

Definition 3.1. (a) An element x of a dcpo P is called *compact* if for every directed subset D of P , $x \leq \bigvee^d D$ implies $x \leq d$ for some $d \in D$. A dcpo P is said to be *compact* if each element of P is compact. An S -dcpo A is called *compact* if it is compact as a dcpo.

(b) An S -dcpo A is called *reversible* if for all $a \in A$ and $s \in S$, there exists $t \in S$ such that $ast = a$.

(c) We denote the full subcategory of **Dcpo** consisted of all compact dcpo's by **C-Dcpo**, and the full subcategory of **Dcpo-S** consisted of all compact S -dcpo's by **C-Dcpo-S**. Also, we denote the full subcategory of all compact reversible S -dcpo's by **C-R-Dcpo-S**.

Compact dcpo's have been called *good* in [16]. In the following proposition, we see a characterization for compact dcpo's.

Proposition 3.2. *For a dcpo P , the following are equivalent:*

- (i) P is a compact dcpo.
- (ii) $\forall D \subseteq^d P$, $\bigvee^d D$ exists in D .
- (iii) Each directed subset of P has a top element.

Proof. (i) $\Rightarrow$ (ii) Let P be a compact dcpo and $D \subseteq^d P$ and $x = \bigvee^d D$. Then, since x is a compact, there exists $d \in D$ where $x = \bigvee^d D \leq d \leq x = \bigvee^d D$. Therefore, $d = \bigvee^d D \in D$. (ii) $\Rightarrow$ (iii) For each directed subset D of P , we have $\bigvee^d D \in D$. Therefore, $\bigvee^d D$ is a top element of D . (iii) $\Rightarrow$ (i) Let $x \in P$ and $D \subseteq^d P$ where $x \leq \bigvee^d D$. Then, since each directed subset

of P has a top element ($\top_D$), we have $\bigvee^d D = \top_D$ and $x \leq \top_D \in D$, as required. $\square$

Example 3.3. Finite posets and Noetherian posets (posets satisfying ascending chain condition on elements) are examples of compact dcpos. Also, for any poset P with discrete order, the posets $P \oplus \top$ and $\perp \oplus P$ are compact dcpos, where $P \oplus \top$ and $\perp \oplus P$ are obtained by adding a top element and a bottom element to P , respectively. Now, let $S = \{s, t, e\}$ where $st = s$ and $ts = s$. Then S with discrete order, is a dcpo-monoid. Take $A = \{\perp, a_1, a_2\}$ where $\perp \leq a_1, a_2$ and $a_1 \parallel a_2$. Consider A with the action $a_1 s = a_2 = a_2 s$, $a_1 t = a_1 = a_2 t$ and $\perp$ is a zero element. Then A is a compact reversible S -dcpo.

It is easy to see the following lemma.

Lemma 3.4. *Let A be a compact reversible S -dcpo with the identity action, B be a compact reversible S -dcpo and $h : A \rightarrow B$ be a monomorphism in $\mathbf{C-R-Dcpo-S}$. Then h is one-one.*

Lemma 3.5. *In $\mathbf{C-R-Dcpo}$ (whenever monomorphisms are one-one), if $f : A \rightarrow B$ is a monomorphism and A is a chain then f is an embedding.*

Proof. Let $f : A \rightarrow B$ be a monomorphism in $\mathbf{C-R-Dcpo}$, A be a chain and $f(a) \leq f(a')$, for some $a, a' \in A$. Then $a \leq a'$ because otherwise $a' < a$, since A is a chain. So $f(a') \leq f(a)$, and hence $f(a) = f(a')$ which implies $a = a'$, because f is one-one. This contradicts $a' < a$. $\square$

Lemma 3.6. *Every non-trivial absolute retract ($\mathcal{E}$ -absolute retract) compact reversible S -dcpo A is bounded by two zero elements.*

Proof. Consider the compact reversible S -dcpo $C = \{\theta_1\} \oplus A \oplus \{\theta_2\}$ obtained by adjoining two zero elements θ_1 and θ_2 to A . Then the retraction $h : C \rightarrow A$ gives the result. $\square$

Corollary 3.7. *Every non-trivial injective ($\mathcal{E}$ -injective) compact reversible S -dcpo A is bounded by two zero elements.*

Theorem 3.8. *The category $\mathbf{C-R-Dcpo-S}$ has no non-trivial injective object.*

Proof. Let A be a non-trivial injective compact reversible S -dcpo. Consider the compact reversible S -dcpo's $A \dot{\cup} \{\theta\}$, where θ is a zero element; and $A \oplus \{\theta\}$, where θ is a zero top element. Define $f : A \dot{\cup} \{\theta\} \rightarrow A \oplus \{\theta\}$ by $f(\theta) = \theta, f(a) = a$, for all $a \in A$. Also, define $g : A \dot{\cup} \{\theta\} \rightarrow A$ by $g(\theta) = \perp_A$ and $g(a) = a$, for all $a \in A$ (by Lemma 3.7, A has a zero bottom element). Now, since A is injective, there exists an S -dcpo map $h : A \oplus \{\theta\} \rightarrow A$ such that $hf = g$. So $h(\theta) = h(f(\theta)) = g(\theta) = \perp_A$ and for every $a \in A$, $h(a) = h(f(a)) = g(a) = a$. Therefore, for every $a \in A$, $a = h(a) \leq h(\theta) = \perp_A$, which contradicts the fact that A is non-trivial. $\square$

By a complete S -dcpo, we mean complete as a poset.

Theorem 3.9. *In $\mathbf{C-R-Dcpo-S}$, an object is $\mathcal{E}$ -injective if it is complete and has the identity action.*

Proof. Let A be a complete compact reversible S -dcpo with the identity action, $i : B \rightarrow C$ be an embedding in $\mathbf{C-R-Dcpo-S}$, and $g : B \rightarrow A$ be an S -dcpo map. Then, define $h : C \rightarrow A$ by $h(x) = \bigvee_{b \in B, b \leq x} g(b)$. The map h preserves the order, because if $x, y \in C$ and $x \leq y$ then $\{g(b) : b \in B, b \leq x\} \subseteq \{g(b) : b \in B, b \leq y\}$ and so $h(x) \leq h(y)$. Also, h is a dcpo map. To see this, let $D \subseteq^d C$. Then $h(D) \subseteq^d A$, since h preserves the order. Also, $h(\bigvee^d D) = \bigvee_{d \in D}^d h(d)$. This is because, $h(\bigvee^d D)$ is an upper bound of $h(D)$. Also $h(\bigvee^d D) \in h(D)$, because C is compact and $\bigvee^d D \in D$. Therefore, it is clear that $h(\bigvee^d D) = \bigvee_{d \in D}^d h(d)$. Now, we prove that h preserves the action. For $x \in C$, we apply the notation $T_x = \{g(b) : b \in B, b \leq x\}$. Let $x \in C$ and $s \in S$. If $T_x = \emptyset$, then $h(x) = \perp_A$. Also, $T_{xs} = \emptyset$, because otherwise, there exists $z = g(b)$ with $b \in B$ and $b \leq xs$. Since C is a reversible S -dcpo, there exists $t \in S$ such that $xst = x$. Therefore $zt = g(bt)$ with $bt \leq xst = x$ and $bt \in B$ which contradicts $T_x = \emptyset$. So, $h(xs) = \perp_A$. On the other hand $h(x)s = \perp_A s = \perp_A$, since $\perp_A$ is a zero element. If $T_x \neq \emptyset$, since the action of A is identity, we have $h(x)s = h(x) = \bigvee T_x$ and $h(xs) = \bigvee T_{xs}$. But,

$T_x = T_{xs}$, for if $z = g(b)$ with $b \in B$ and $b \leq x$, then $z = g(b)s = g(bs)$ (the first equality is because, $g(b) \in A$ and the action of A is identity) and $bs \leq xs$. Also, if $z = g(b)$ with $b \in B$ and $b \leq xs$ then, since C is reversible, there exists $t \in S$ with $xst = x$ and hence $z = g(b) = g(b)t = g(bt)$ and $bt \leq xst = x$. Therefore, $h(xs) = h(x)s$. $\square$

As a consequence of Theorem 3.9 we have:

Corollary 3.10. *In $\mathbf{C-R-Dcpo-S}$, an object is $\mathcal{E}$ -absolute retract if it is complete and has the identity action.*

Theorem 3.11. *In $\mathbf{C-R-Dcpo-S}$, an object is absolute retract if it is a complete compact chain with the identity action.*

Proof. Let A be a complete compact chain with the identity action. Then, by Corollary 3.10, A is $\mathcal{E}$ -absolute retract. Now, let $f : A \rightarrow C$ be a monomorphism in $\mathbf{C-R-Dcpo-S}$. Then, by Lemmas 3.4 and 3.5, f is an embedding in $\mathbf{C-R-Dcpo-S}$. Therefore, there exists an S -dcpo map $h : C \rightarrow A$ such that $hf = id_A$. $\square$

Applying Theorems 3.8 and 3.11, we get that injectivity in $\mathbf{C-R-Dcpo-S}$ is not the same as absolute retractness. In the following theorem, we see that the action of $\mathcal{E}$ -absolute retract object in $\mathbf{C-R-Dcpo-S}$, need not be identity.

Theorem 3.12. *In $\mathbf{C-R-Dcpo-S}$, $M_n = \perp \oplus \{m : 1 \leq m \leq n\} \oplus \top$, for all $n \in \mathbb{N}^\infty$ with $n \geq 2$ (where $\{m : 1 \leq m \leq n\}$ has been considered with discrete order) with zero top and bottom elements is $\mathcal{E}$ -absolute retract.*

Proof. Let $i : M_n \hookrightarrow C$ be an embedding in $\mathbf{C-R-Dcpo-S}$. Define $h : C \rightarrow M_n$ by

$$h(x) = \begin{cases} x & \text{if } x \in M_n \\ \top & \text{if } x \notin M_n, y \leq x, \perp \neq y \in M_n \\ \perp & \text{otherwise} \end{cases}$$

To prove that h preserves the order, let $x_1 \leq x_2$ in C . If $x_1, x_2 \in M_n$ then it is clear that $h(x_1) \leq h(x_2)$. If $x_1, x_2 \notin M_n$ then two cases may occur. If there exists $\perp \neq y \in M_n$ where $y \leq x_1 \leq x_2$, then $h(x_1) = h(x_2) = \top$. Otherwise, if there does not exist $\perp \neq y \in M_n$ where $y \leq x_1$ then $\perp = h(x_1) \leq h(x_2)$. Now, if $\perp \neq x_1 \in M_n$ and $x_2 \notin M_n$ then we have $x_1 = h(x_1) \leq h(x_2) = \top$. Also, if $x_1 = \perp \in M_n$ and $x_2 \notin M_n$, then $\perp = h(x_1) \leq h(x_2)$. Finally, if $x_1 \notin M_n$ and $x_2 \in M_n$ then two cases may occur. If $x_2 = \top$ then clearly $h(x_1) \leq h(x_2) = \top$. Otherwise, by the definition of order in M_n , there does not exist $\perp \neq y \in M_n$ where $y \leq x_1$. So $\perp = h(x_1) \leq h(x_2)$.

To show that h is continuous, let $D \subseteq^d C$. Since h preserves the order, we get $h(D) \subseteq^d M_n$ and $h(\bigvee^d D)$ is an upper bound of $h(D)$. Also $h(\bigvee^d D) \in h(D)$, because C is compact and $\bigvee^d D \in D$. Therefore, $h(\bigvee^d D) = \bigvee_{d \in D}^d h(d)$.

To prove that h preserves the action, let $x \in C$ and $s \in S$. If $x \in M_n$ then $xs \in M_n$ and $h(xs) = xs = h(x)s$. If $x \notin M_n$ then $xs \notin M_n$. This is because, on the contrary, if $xs \in M_n$ then, since C is reversible, there exists $t \in S$ where $x = xst \in M_n$, which contradicts $x \notin M_n$. Now, if there exists $\perp \neq y \in M_n$ such that $y \leq x$ then $ys \leq xs$, $ys \in M_n$ and $ys \neq \perp$. The latter is because, on the contrary, if $ys = \perp$, then, since C is reversible, there exists $t' \in S$ such that $y = yst' = \perp t' = \perp$, since $\perp$ is zero. It contradicts $y \neq \perp$. So $xs \notin M_n$ and there exists $ys \in M_n$ where $ys \neq \perp$ and $ys \leq xs$. Therefore, $h(xs) = \top = \top s = h(x)s$. Otherwise, if there does not exist $\perp \neq y \in M_n$ such that $y \leq x$, there does not exist $\perp \neq y' \in M_n$ such that $y' \leq xs$. This is because, on the contrary, if there exists $\perp \neq y' \in M_n$ such that $y' \leq xs$, then, since C is reversible, there exists $t \in S$ where $y't \leq xst = x$, $y't \in M_n$ and $y't \neq \perp$ (on the contrary, if $y't = \perp$ then, there exists $t'' \in S$ where $y' = y'tt'' = \perp t'' = \perp$, which contradicts $y' \neq \perp$). Hence $h(xs) = \perp = \perp s = h(x)s$, and the proof is complete. $\square$

Theorem 3.13. *Let A be a compact reversible S -dcpo with zero top and bottom elements. Also, let there exist a unique zero element $a \in A$ where for every $\perp \neq x_1, \top \neq x_2 \in A$ such that $x_1 \leq x_2$, we have $x_1 \leq a \leq x_2$.*

Then A is $\mathcal{E}$ -absolute retract in **C-R-Dcpo-S**.

Proof. Let $i : A \hookrightarrow C$ be an embedding in **C-R-Dcpo-S**. Define $h : C \rightarrow A$ by

$$h(x) = \begin{cases} x & \text{if } x \in A \\ \top & \text{if } \exists x_1 \in A, x_1 \leq x \notin A, \nexists x_2 \in A, x_2 \neq \top, x \leq x_2 \\ a & \text{if } x \notin A, \exists x_1, x_2 \in A, \perp \neq x_1 \leq x \leq x_2 \neq \top \\ \perp & \text{o.w} \end{cases}$$

To show that h is an S -dcpo map, first we prove that h preserves the order. Let $x \leq x'$ in C . If $x, x' \in A$ then $h(x) \leq h(x')$. If $x, x' \notin A$ then three cases may occur.

Case (i): There exists $x_1 \in A$ where $\perp \neq x_1$ and $x_1 \leq x$. We have $x_1 \leq x \leq x'$. Now, if there does not exist $x_2 \in A$ such that $x_2 \neq \top$ and $x \leq x_2$, then there does not exist $x_2 \in A$ such that $x_2 \neq \top$ and $x' \leq x_2$, since $x \leq x'$. So $h(x) = h(x') = \top$. Otherwise, if there exists $x_2 \in A$ such that $x_2 \neq \top$ and $x \leq x_2$ then $h(x) = a$. On the other hand, either there exists $x_3 \in A$ such that $x_3 \neq \top$, $x' \leq x_3$ and so $h(x') = a$ or there does not exist $x_3 \in A$ such that $x_3 \neq \top$, $x' \leq x_3$ and hence $h(x') = \top$. In any case $h(x) \leq h(x')$.

Case (ii): $\perp \leq x_1$. We have $\perp \leq x \leq x'$. Now, if there does not exist $x_2 \in A$ such that $x_2 \neq \top$ and $x \leq x_2$, then there does not exist $x_2 \in A$ such that $x_2 \neq \top$ and $x' \leq x_2$, since $x \leq x'$. So $h(x) = h(x') = \top$. Otherwise, if there exists $x_2 \in A$ such that $\top \neq x_2$ and $x \leq x_2$ then $h(x) = \perp$. So $\perp = h(x) \leq h(x')$.

Case (iii): There does not exist $x_1 \in A$ where $x_1 \leq x$. In this case, $h(x) = \perp$ and clearly $\perp = h(x) \leq h(x')$.

Another case, if $x \in A$, $x \neq \perp$ and $x' \notin A$, then $h(x) = x$. Now, if there exists $x_2 \in A$ where $x_2 \neq \top$ and $x' \leq x_2$, then $x = h(x) \leq h(x') = a$. The latter is because, by the assumption $x \leq a \leq x_2$. Otherwise, if there does not exist $x_2 \in A$ where $x_2 \neq \top$ and $x' \leq x_2$, then $x = h(x) \leq h(x') = \top$. Also, if $x = \perp \in A$ and $x' \notin A$ then $\perp = h(x) \leq h(x')$. Finally, if $x \notin A$,

$x' \in A$ and $x' \neq \top$, then two cases may occur. If there exists $x_1 \in A$ where $x_1 \neq \perp$ and $x_1 \leq x$ then $h(x) = a \leq x' = h(x')$. The latter is because, by the assumption $x_1 \leq a \leq x'$. Otherwise, if there does not exist $x_1 \in A$ where $x_1 \neq \perp$ and $x_1 \leq x$, then $\perp = h(x) \leq h(x') = x'$. Also, in the case where $x \notin A$ and $x' = \top \in A$, it is clear that $h(x) \leq h(x') = \top$.

To show that h is continuous, let $D \subseteq^d C$. Since h preserves the order, we get $h(D) \subseteq^d A$ and $h(\bigvee^d D)$ is an upper bound of $h(D)$. Also $h(\bigvee^d D) \in h(D)$, because C is compact and $\bigvee^d D \in D$. Therefore, $h(\bigvee^d D) = \bigvee_{d \in D}^d h(d)$.

To prove that h preserves the action, let $x \in C$ and $s \in S$. If $x \in A$ then $xs \in A$ and $h(xs) = xs = h(x)s$. If $x \notin A$ then $xs \notin A$. This is because, on the contrary, if $xs \in A$ then, since C is a reversible, there exists $t \in S$ where $x = xst \in A$. Now, if there exists $x_1 \in A$ such that $x_1 \leq x$ and there does not exist $x_2 \in A$ where $x_2 \neq \top$ and $x \leq x_2$, then $x_1s \leq xs$, $x_1s \in A$ and there does not exist $x_3 \in A$ where $x_3 \neq \top$ and $xs \leq x_3$. The latter is because there exists $t \in S$ such that $x = xst \leq x_3t$, $x_3t \in A$ and $x_3t \neq \top$ (on the contrary, if $x_3t = \top$ then, since C is a reversible, there exists $t' \in S$ such that $x_3 = x_3tt' = \top t' = \top$, which is a contradiction. Therefore $h(xs) = \top = \top s = h(x)s$. If there exist $\perp \neq x_1$ and $\top \neq x_2$ where $x_1 \leq x \leq x_2$, then $x_1s \leq xs \leq x_2s$, $x_1s, x_2s \in A$, $x_1s \neq \perp$ and $x_2s \neq \top$. This is because there exists $s', s'' \in S$ where $x_1 = x_1ss' = \perp s' = \perp$ and $x_2 = x_2ss'' = \top s'' = \top$, which are contradictions. Hence, $h(xs) = a = as = h(x)s$. Finally, if x does not satisfy any of these conditions then xs does not satisfy any of these conditions, too. This is because, on the contrary, if there exists $x_1 \in A$ such that $x_1 \leq xs$ and there does not exist $x_2 \in A$ where $x_2 \neq \top$ and $xs \leq x_2$, then, there exists $t \in S$ where $x_1t \leq xst = x$ which contradicts the fact that there does not exist $y \in A$ such that $y \leq x$. Also, on the contrary, if there exist $\perp \neq x_1$, $\top \neq x_2 \in A$ such that $x_1 \leq xs \leq x_2$ then, there exists $t \in S$ where $x_1t \leq xst = x \leq x_2t$, $x_1t, x_2t \in A$, $x_1t \neq \perp$ and $x_2t \neq \top$, which contradicts the fact that there does not exist $\perp \neq a_1$, $\top \neq a_2 \in A$ such that $a_1 \leq x \leq a_2$. So $h(xs) = \perp = \perp s = h(x)s$. $\square$

4 Retractness of compact strongly reversible dcpo acts

In this section we study injectivity and absolute retracts in the category of compact strongly reversible directed complete poset acts and we show that injectivity does not coincide with retracts in this category, too. Also, we characterize $\mathcal{E}$ -injective objects.

Definition 4.1. An S -dcpo A is called *strongly reversible* if for all $a \in A$ and $s \in S$, we have $as^2 = a$. We denote the full subcategory of all compact strongly reversible S -dcpo's by **C-SR-Dcpo- S** .

Example 4.2. Let $S = \{s, t, e\}$ where $st = s = ts$ and $t^2 = t = s^2$. Then, S with discrete order is a dcpo-monoid. Now, consider $A = \{a_1, a_2\}$ where $a_1 \parallel a_2$ with the action $a_1s = a_2 = a_2t$ and $a_2s = a_1 = a_1t$. Then A is a compact strongly reversible S -dcpo.

The counterparts of Lemmas 3.4, 3.5 and Corollary 3.7 hold **C-SR-Dcpo- S** .

Theorem 4.3. *The category **C-SR-Dcpo- S** has no non-trivial injective object.*

Proof. Let A be a non-trivial injective compact strongly reversible S -dcpo. Consider the compact strongly reversible S -dcpo's $A \dot{\cup} \{\theta\}$, where θ is a zero element; and $A \oplus \{\theta\}$, where θ is a zero top element. Define $f : A \dot{\cup} \{\theta\} \rightarrow A \oplus \{\theta\}$ by $f(\theta) = \theta, f(a) = a$, for all $a \in A$. Also, define $g : A \dot{\cup} \{\theta\} \rightarrow A$ by $g(\theta) = \perp_A$ and $g(a) = a$, for all $a \in A$ (by Lemma 3.7, A has a zero bottom element). Now, since A is injective, there exists an S -dcpo map $h : A \oplus \{\theta\} \rightarrow A$ such that $hf = g$. So $h(\theta) = h(f(\theta)) = g(\theta) = \perp_A$ and for every $a \in A$, $h(a) = h(f(a)) = g(a) = a$. Therefore, for every $a \in A$, $a = h(a) \leq h(\theta) = \perp_A$, which contradicts the fact that A is non-trivial. $\square$

Recall that an S -poset A is *equivariantly complete* if it is complete and $\bigvee(Xs) = (\bigvee X)s$, for all $s \in S$ and $X \subseteq A$.

Lemma 4.4. *Let A be a complete strongly reversible S -dcpo. Then A is equivariantly complete.*

Proof. Let A be a complete strongly reversible S -dcpo, $X \subseteq A$ and $s \in S$. Then $xs \leq (\bigvee X)s$, for all $x \in X$. If a is another upper bound of Xs in A , then for all $x \in X$, we get $x = xs^2 \leq as$. So $\bigvee X \leq as$ and $(\bigvee X)s \leq as^2 = a$. Therefore, $\bigvee (Xs) = (\bigvee X)s$, for all $s \in S$. $\square$

Theorem 4.5. *In $\mathbf{C-SR-Dcpo-S}$, an object A is $\mathcal{E}$ -injective if it is complete.*

Proof. Let $i : B \rightarrow C$ be an embedding in $\mathbf{C-SR-Dcpo-S}$ and $g : B \rightarrow A$ be an S -dcpo map. Then, define $h : C \rightarrow A$ by $h(x) = \bigvee_{b \in B, b \leq x} g(b)$. The map h preserves the order because if $x \leq y$ in C then $\{g(b) : b \in B, b \leq x\} \subseteq \{g(b) : b \in B, b \leq y\}$ and so $h(x) \leq h(y)$. Also, h is a dcpo map. To see this, let $D \subseteq^d C$. Then $h(D) \subseteq^d A$, since h preserves the order. Also, $h(\bigvee^d D) = \bigvee_{d \in D} h(d)$. This is because, $h(\bigvee^d D)$ is an upper bound of $h(D)$. Also $h(\bigvee^d D) \in h(D)$, because C is compact and $\bigvee^d D \in D$. Therefore, it is clear that $h(\bigvee^d D) = \bigvee_{d \in D} h(d)$.

Now, we prove that h preserves the action. For $x \in C$, let $T_x = \{g(b) : b \in B, b \leq x\}$. Let $x \in C$ and $s \in S$. If $T_x = \emptyset$, then $h(x) = \perp_A$. Also, $T_{xs} = \emptyset$, because otherwise, there exists $z = g(b)$ with $b \in B$ and $b \leq xs$, then $z = zs^2 = g(bs^2)$ with $bs^2 \leq xs^3 = xs$ and $bs^2 \in B$, which contradicts $T_x = \emptyset$. So, $h(xs) = \perp_A$. On the other hand, $h(xs) = h(x)s = \perp_A s = \perp_A$, since $\perp_A$ is zero. This is because $\perp_A \leq \perp_A s$. So $\perp_A s \leq \perp_A s^2 = \perp_A$. If $T_x \neq \emptyset$, by Lemma 4.4, we have

$$h(x)s = (\bigvee_{b \in B, b \leq x} g(b))s = \bigvee_{b \in B, b \leq x} (g(b)s) = \bigvee_{b \in B, b \leq x} g(bs).$$

On the other hand, $h(xs) = \bigvee_{b \in B, b \leq xs} g(b)$. But, $\{g(b) : b \leq xs\} = \{g(bs) : b \leq x\}$, because if $z = g(b)$ with $b \leq xs$ then $z = g((bs)s)$ and $bs \leq xs^2 = x$; also if $z = g(bs)$ with $b \leq x$ then $z = g(bs)$ and $bs \leq xs$. Therefore, $h(xs) = h(x)s$ and the proof is complete. $\square$

As a consequence of Theorem 4.5 we have:

Corollary 4.6. *In $\mathbf{C-SR-Dcpo-S}$, an object is $\mathcal{E}$ -absolute retract if it is complete.*

Theorem 4.7. *In $\mathbf{C-SR-Dcpo-S}$, an object is absolute retract if it is a complete compact chain with the identity action.*

Proof. Let A be a complete compact chain with the identity action. Then, by Corollary 4.6, A is $\mathcal{E}$ -absolute retract. Now, let $f : A \rightarrow C$ be a monomorphism in $\mathbf{C-SR-Dcpo-S}$. Then, f is an embedding in $\mathbf{C-SR-Dcpo-S}$. Therefore, there exists an S -dcpo map $h : C \rightarrow A$ such that $hf = id_A$. $\square$

Applying Theorems 4.3 and 4.7, we get that injectivity in $\mathbf{C-SR-Dcpo-S}$ is not the same as retractness.

In what follows, we continue the study of retracness using internal order sum.

Definition 4.8. A poset P is said to be *order sum* of posets P' and P'' if for every $x \in P'$ and $x' \in P''$, $x \leq x'$ and we denote it by $P = P' \oplus P''$. An S -dcpo A is an *internal order sum* of its sub S -dcpo's A_1 and A_2 , if $A = A_2 \oplus A_1$, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$.

Theorem 4.9. *Let A be a compact reversible S -dcpo and A be an internal order sum of its sub S -dcpo's A_1 and A_2 . If A_1 and A_2 are $\mathcal{E}$ -absolute retract in $\mathbf{C-R-Dcpo-S}$, then A is $\mathcal{E}$ -absolute retract in $\mathbf{C-R-Dcpo-S}$.*

Proof. Since A_1 and A_2 are $\mathcal{E}$ -absolute retract and C is an $\mathcal{E}$ -extension of A_1 and A_2 , there exist retractions $h_1 : C \rightarrow A_1$ and $h_2 : C \rightarrow A_2$. Now, define $h : C \rightarrow A$ where:

$$h(x) = \begin{cases} x & \text{if } x \in A \\ h_2(x) & \text{if } x \notin A, \exists b \in A_2, x \leq b \\ h_1(x) & \text{otherwise} \end{cases}$$

We show that h is an S -dcpo map. First, we prove that h preserves the order. To see this, let $x_1, x_2 \in C$ such that $x_1 \leq x_2$. If $x_1, x_2 \in A$ then, by the definition of h , it is clear that $h(x_1) \leq h(x_2)$. Now, suppose that $x_1, x_2 \notin A$. If there exists $b \in A_2$ such that $x_1 \leq b$, then $h(x_1) = h_2(x_1)$ and $h(x_2) = h_1(x_2)$ or $h(x_2) = h_2(x_2)$. If the first case occur, then $h(x_1) = h_2(x_1) \leq h_1(x_2) = h(x_2)$, since $A = A_2 \oplus A_1$, $h_2(x_1) \in A_2$ and $h_1(x_2) \in A_1$. Also in the second case, it is clear that $h(x_1) \leq h(x_2)$. Otherwise, if there does not exist $b \in A_2$ with $x_1 \leq b$, then $h(x_1) = h_1(x_1)$ and we have there does not exist $b' \in A_2$ where $x_2 \leq b'$. This is because, on the contrary, if there exists $b' \in A_2$ where $x_2 \leq b'$ then $x_1 \leq x_2 \leq b' \in A_2$ which is a contradiction. So, $h(x_2) = h_1(x_2)$ and $h(x_1) \leq h(x_2)$. Now, if $x_1 \notin A$ and $x_2 \in A$ then two cases may occur. If $x_2 \in A_1$ then $h(x_2) = x_2 = h_1(x_2)$. The latter is because, A_1 is a retract of C and $h_1(a) = a$ for all $a \in A_1$. On the other hand, $h(x_1) = h_1(x_1)$ or $h(x_1) = h_2(x_1)$ and it is straightforward to show that $h(x_1) \leq h(x_2)$. In the second case, if $x_2 \in A_2$ then $h(x_1) = h_2(x_1)$, since $x_1 \leq x_2 \in A_2$. Also, $h(x_2) = x_2 = h_2(x_2)$, since A_2 is a retract of C and $h_2(a) = a$ for all $a \in A_2$. So, it is clear that $h(x_1) \leq h(x_2)$. Finally, if $x_1 \in A_1$ and $x_2 \notin A$ then $h(x_1) = x_1 = h_1(x_1)$ and $h(x_2) = h_1(x_2)$. The latter is because $x_1 \in A_1$, $x_1 \leq x_2$ and $A = A_2 \oplus A_1$. Therefore, there does not exist $b \in A_2$ such that $x_2 \leq b$. So $h(x_1) \leq h(x_2)$. Also, if $x_1 \in A_2$ and $x_2 \notin A$ then $h(x_1) = x_1 = h_2(x_1)$ and $h(x_2) = h_1(x_2)$ or $h(x_2) = h_2(x_2)$. Hence, it is clear that $h(x_1) \leq h(x_2)$. So, h preserves the order.

Now, we show that h is continuous. Let $D \subseteq^d C$. Since h preserves the order, we get $h(D) \subseteq^d M_n$ and $h(\bigvee^d D)$ is an upper bound of $h(D)$. Also $h(\bigvee^d D) \in h(D)$, because C is compact and $\bigvee^d D \in D$. Therefore, it is clear that $h(\bigvee^d D) = \bigvee_{d \in D}^d h(d)$.

Now, we show that h preserves the action. To see this, let $x \in C$ and $s \in S$. If $x \in A$ then $xs \in A$ and it is clear that $h(xs) = h(x)s$. Now, let $x \in C \setminus A$ and $s \in S$. So $xs \in C \setminus A$. The latter is because, on the contrary, if $xs \in A$ then, since C is reversible, there exists $t \in S$ such that $xst = x \in A$, which contradicts $x \in C \setminus A$. Now, if there exists an element $b \in A_2$, where $x \leq b$, then $xs \leq bs$ and $bs \in A_2$. Therefore,

$h(xs) = h_2(xs) = h_2(x)s = h(x)s$. On the other hand, if there does not exist an element $b \in A_2$ where $x \leq b$, then there does not exist an element $b' \in A_2$ such that $xs \leq b'$. This is because, on the contrary, if there exists an element $b' \in A_2$ such that $xs \leq b'$, then again, since C is reversible, there exists $t \in S$ where $x = xst \leq b't \in A_2$, which is a contradiction. Hence $h(xs) = h_1(xs) = h_1(x)s = h(x)s$. $\square$

In the following, we see that the converse of Theorem 4.9 is not true.

Remark 4.10. By Theorem 3.12, $M_2 = \{\perp, a_1, a_2, \top\}$ where $a_1 \parallel a_2$ with the identity action is $\mathcal{E}$ -absolute retract in **C-R-Dcpo-S**. Take $A_2 = \{\perp, a_1, a_2\}$ and $A_1 = \{\top\}$. It is clear that M_2 is an internal order sum of A_1 and A_2 , but by Lemma 3.6, A_2 is not $\mathcal{E}$ -absolute retract in **C-R-Dcpo-S**. The latter is because, A_2 does not have a zero top element.

Conclusion

Actions of a monoid or a group on a set have always been of interest to mathematicians and computer scientists. Also, the algebraic theory of a particular type of automata is nothing but the action of a free semigroup on a set of states. On the other hand, domain theory, which studies directed complete partially ordered sets (dcpo's), is a field on the borderline between mathematics and computer science.

The notion of retractness is a crucial notion in many branches of mathematics and is also tightly connected to injectivity. For example, injective T_0 spaces are exactly the retracts of the powers of Sierpinski space.

In this paper, we considered actions of a semigroup on compact reversible dcpo's and studied retractness in the category so obtained. We have shown, among other things, that a compact reversible dcpo act is absolute retract, with respect to embeddings, if it is complete and has the identity action, and it is absolute retract, with respect to monomorphisms, if it is a complete compact chain with the identity action.

We have also seen that the category of compact strongly reversible dcpo acts has no non-trivial injective, with respect to monomorphisms, but every complete one is injective with respect to embeddings.

Finally, we studied retractness using order sum and have seen that a compact reversible S -dcpo which is an internal order sum of absolute retract subobjects A_1 and A_2 is itself absolute retract, with respect to embeddings.

Acknowledgement

The authors would like to thank very much the editor and the referee for their comments, for pointing some mistakes and adding an informative part to make the paper better understandable.

References

- [1] Abramsky, S.: *Domain theory in logical form*, Ann. Pure Appl. Logic 51 (1991), 1-77.
- [2] Abramsky, S. and A. Jung: "Domain Theory". In: Abramsky, S., Gabbay, D. M. and Maibaum, T. S. E. (editors), Handbook of Logic in Computer Science 3, Clarendon Press, Oxford, 1995.
- [3] Banaschewski, B.: *Injectivity and essential extensions in equational classes of algebras*, Queen's Paper Pure Appl. Math. 25 (1970), 131-147.
- [4] Bulman-Fleming S., and M. Mahmoudi: *The category of S-posets*, Semigroup Forum 71(3) (2005), 443-461.
- [5] Cagliari, F. and S. Mantovani: *Injectivity and sections*, J. Pure Appl. Algebra 204 (2006), 79-89.
- [6] Davey, B.A. and H.H. Priestly: "Introduction to Lattices and Order", Cambridge University Press, Cambridge, 1990.
- [7] Ebrahimi, M.M.: *Algebra in a topos of sheaves: Injectivity in quasi-equational classes*, J. Pure Appl. Alg. 26 (1982), 269-280.
- [8] Ebrahimi, M.M.: *Internal completeness and injectivity of Boolean algebras in the topos of M-set*, Bull. Austral. Math. 41(2) (1990), 323-332.
- [9] Ebrahimi, M.M. and M. Mahmoudi: *The category of M-sets*, Italian J. Pure Appl. Math. 9 (2001), 123-132.
- [10] Ebrahimi, M.M. and M. Mahmoudi: *On injectivity of projection and separated projection algebras*, Italian J. Pure Appl. Math. 29 (2012), 43-54.

- [11] Ebrahimi, M.M., H. Haddadi, and M. Mahmoudi: *Injectivity in a category: An overview of well behaviour theorems*, Algebra, Groups, and Geometries, 26 (2009), 451-472.
- [12] Ebrahimi, M. M., M. Mahmoudi, and H. Rasouli; *Banaschewski's Theorem for S -posets: Regular injectivity and Completeness*, Semigroup Forum 80(2) (2010), 313-324.
- [13] Kilp, M., U. Knauer, and A. Mikhalev: "Monoids, Acts and Categories", Walter de Gruyter, Berlin, New York, 2000.
- [14] Mahmoudi, M., *Internal injectivity of Boolean algebras in $\mathbf{MSet}$* , Algebra Universalis 41(3) (1999), 155-175.
- [15] Mahmoudi, M. and H. Moghbeli: *The category of monoid actions in $\mathbf{Cpo}$* , Bull. Iranian Math. Soc. 41(1) (2015), 159-175.
- [16] Mahmoudi, M. and H. Moghbeli: *Free and cofree acts of dcpo-monoids on directed complete posets*, to appear in the Bull. Malays. Math. Sci. Soc. (Available online at: <http://www.emis.de/journals/BMMSS/pdf/acceptedpapers/2013-03-007-R1.pdf>).
- [17] Mahmoudi, M., M. Yavari, and M.M. Ebrahimi: *On injectivity of automata on directed complete posets*, submitted.

**ON THE PLANAR AND OUTER PLANAR
ANNIHILATING-IDEAL GRAPHS OF A LATTICE**

F. Shahsavar*

Department of Pure Mathematics
International Campus of Ferdowsi
University of Mashhad
P.O. Box 1159-91775, Mashhad, Iran
fa.shahsavar@yahoo.com

Received May 2, 2015

Abstract

Let $(L, \wedge, \vee)$ be a lattice with a least element 0. The annihilating-ideal graph of L , denoted by $\mathcal{AG}(L)$, is a graph whose vertex-set is the set of all nontrivial ideals of L and, for every two distinct vertices I and J , I is adjacent to J if and only if $I \wedge J = \{0\}$. In this paper, we completely determine when the annihilating-ideal graph is planar or outer planar.

AMS Classification 05C10, 06A07

Keywords. Lattice, Annihilating-ideal graph, Planar graph, outer planar graph.

*Corresponding author

1 Introduction

The study of algebraic structures, using the properties of graphs, has become an exciting research topic in the last twenty years. Several classes of graphs associated with algebraic structures have been actively investigated. For example, Cayley graphs have been studied in [12] and [13], zero-divisor graphs have been studied in [3], [4], [5], [10], and cozero-divisor graphs have been introduced in [2].

Let R be a commutative ring, $\mathbb{I}(R)$ be the set of all ideals of R . In [6] and [7], Behboodi and Rakeei defined and studied the annihilating-ideal graph of R , denoted by $\mathbb{AG}(R)$. Let $\mathbb{A}(R)$ be the set of annihilating-ideals of R , where a non-zero ideal I of R is called an *annihilating-ideal*, if there exists a non-zero ideal J of R such that $IJ = 0$. The annihilating-ideal graph of R is a simple graph with vertex-set $\mathbb{A}(R)$ such that two distinct vertices I and J are adjacent, denoted by $I - J$, if and only if $IJ = 0$.

A *Lattice* is an algebra $L = (L, \wedge, \vee)$ satisfying the following conditions: for all $a, b, c \in L$,

- (i) $a \wedge a = a, a \vee a = a$,
- (ii) $a \wedge b = b \wedge a, a \vee b = b \vee a$,
- (iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c), a \vee (b \vee c) = (a \vee b) \vee c$, and
- (iv) $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

Note that in every lattice the equality $a \wedge b = a$ always implies that $a \vee b = b$. By [14, Theorem 2.1], one can define an order $\leq$ on a lattice L as follows:

For any $a, b \in L$, we set $a \leq b$ if and only if $a \wedge b = a$.

Then $(L, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, $\text{g.l.b.}(a, b)$, $\text{l.u.b.}(a, b) \in P$. For each a and b in P , we define $a \wedge b := \text{g.l.b.}(a, b)$ and $a \vee b := \text{l.u.b.}(a, b)$. Then $(P, \wedge, \vee)$ is a lattice.

Let $(L, \wedge, \vee)$ be a lattice with a least element 0 and I be a non-empty subset of L . We say that I is an *ideal* of L , denoted by $I \trianglelefteq L$, if

- (i) for all $a, b \in L$, $a \vee b \in L$.
- (ii) If $0 \leq a \leq b$ and $b \in I$, then $a \in I$.

For two distinct ideals I and J of a lattice L we put

$$I \wedge J := \{x \wedge y \ ; \ x \in I, y \in J\}.$$

The annihilating-ideal graph of L , denoted by $\mathbb{AG}(L)$, is defined by present authors in [1]. $\mathbb{AG}(L)$ is a graph whose vertex-set is the set of all non-trivial ideals of L and, for every two distinct vertices I and J , I is adjacent to J , if and only if $I \wedge J = 0$.

In this paper, we study the planarity and outer planarity of the annihilating-ideal graphs of lattices.

Now, we recall some definitions and notations on graphs and lattices. We use the standard terminology of graphs in [8] and lattices in [9].

In a graph G , the *distance* between two distinct vertices a and b , denoted by $d(a, b)$, is the length of the shortest path connecting a and b , if such a path exists; otherwise, we set $d(a, b) := \infty$. The *diameter* of a graph G is $\text{diam}(G) = \sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. For a positive integer r , an *r -partite graph* is one whose vertex set can be partitioned into r subsets, so that no edges has both ends in any one subset. A *complete r -partite graph* is one in which each vertex is joined to every vertex that is not in the same part. In the graph theory, a *unicycle graph* is a graph that has exactly one cycle.

In a lattice $(L, \wedge, \vee)$ with a least element 0 , an element a is called an *atom* if $a \neq 0$ and, for an element x in L , the relation $0 \leq x \leq a$ implies that either $x = 0$ or $x = a$. We denote the set of all atoms of L by $A(L)$, and, for an ideal I of L , $A(I)$ denotes the set of all atoms contained in I .

In this paper, we assume that L is a finite lattice and $A(L) = \{a_1, a_2, \dots, a_n\}$ is the set of all atoms of L .

2 Planarity of $\mathbb{AG}(L)$

In this section, we completely characterize all lattices L such that $\mathbb{AG}(L)$ is planar. We begin this section with the following notation.

Notation 2.1. Let $i_1, i_2, \dots, i_k$ be integers with $1 \leq i_1 < i_2 < \dots < i_k \leq n$. The notation $U_{i_1 i_2 \dots i_k}$ stands for the following set:

$$\{I \subseteq L; \quad \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \subseteq I \quad \text{and} \quad a_j \notin I, \text{ for } j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}\}.$$

Note that no two distinct elements in $U_{i_1 i_2 \dots i_k}$ are adjacent in $\mathbb{AG}(L)$. Also if the index sets $\{i_1, i_2, \dots, i_k\}$ and $\{j_1, j_2, \dots, j_{k'}\}$ of $U_{i_1 i_2 \dots i_k}$ and $U_{j_1 j_2 \dots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \dots i_k} \cap U_{j_1 j_2 \dots j_{k'}} = \emptyset$. Moreover, $V(\mathbb{AG}(L)) = \bigcup U_{i_1 i_2 \dots i_k}$, for all $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Also by $1 \dots \hat{i} \dots n$ we mean that $1 \dots i-1 \ i+1 \dots n$. Note that $U_{1 \dots n}$ consist of isolated vertices.

A graph G is said to be *planar* if it can be drawn in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Clearly, isolated points do not affect planarity. Hence, we ignore the set $U_{12 \dots n}$ from the vertex-set of $\mathbb{AG}(L)$, and so, we do not show these points in our figures.

The following lemma is useful.

Lemma 2.2. If $\mathbb{AG}(L)$ is planar, then $|A(L)| \leq 4$.

Proof. Suppose on the contrary that $|A(L)| > 4$. Since the induced subgraph of $\mathbb{AG}(L)$ on vertex-set $\{\{0, a_i\}\}$, for $1 \leq i \leq n$, is a complete graph, one can find a subgraph isomorphic to K_5 and therefore, by Kuratowski's Theorem, $\mathbb{AG}(L)$ is not planar. Hence we have $|A(L)| \leq 4$. $\square$

By Lemma 2.2, we study the cases that $|A(L)|$ is equal to 1, 2, 3 or 4. If $|A(L)| = 1$, then $\mathbb{AG}(L)$ is a totally disconnected graph, and so it is planar. In the following proposition we state a necessary and sufficient condition for the planarity of $\mathbb{AG}(L)$, when $|A(L)| = 2$.

Proposition 2.3. *Suppose that $|A(L)| = 2$. Then $\mathbb{AG}(L)$ is planar if and only if $|U_1| \leq 2$ or $|U_2| \leq 2$.*

Proof. Let $\mathbb{AG}(L)$ be planar. Assume on the contrary that $|U_i| \geq 3$, for all $i = 1, 2$. Clearly all vertices in U_1 are adjacent to all vertices in U_2 , and so $\mathbb{AG}(L)$ has a subgraph isomorphic to $K_{3,3}$ with vertex-set $U_1 \cup U_2$, a contradiction.

The converse statement is clear, since $\mathbb{AG}(L)$ is a complete bipartite graph with parts U_1 and U_2 . $\square$

In the sequel of this section, we investigate the planarity of $\mathbb{AG}(L)$, when $|A(L)| = 3$.

If $|\bigcup_{i=1}^3 U_i| \geq 5$ and $|U_i| = |U_j| = 1$, for some $1 \leq i \neq j \leq 3$, then we have the following two cases:

Case 1. $U_{ij} = \emptyset$, for some $1 \leq i \neq j \leq 3$. Then $\mathbb{AG}(L)$ is isomorphic to the graph of Figure 1, and so $\mathbb{AG}(L)$ is planar.

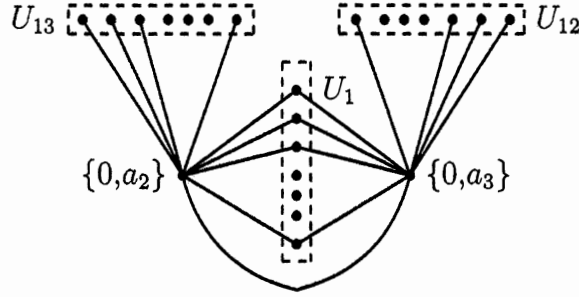


Figure 1:

Case 2. $U_{ij} \neq \emptyset$, for some $1 \leq i \neq j \leq 3$, and one of the U_i 's, say U_1 , has three elements and also if $U_{23} \neq \emptyset$, then U_1 and $U_{23} \cup U_2 \cup U_3$ induce $K_{3,3}$. Therefore $\mathbb{AG}(L)$ is not planar.

If $|\bigcup_{i=1}^3 U_i| \geq 7$ and at least for two of U_i 's, $|U_i| \geq 2$, then one of the U_i 's has at least 3 elements, say U_1 . So there exist two distinct ideals I_1 and I_2 such that $I_1, I_2 \in U_1 \setminus \{0, a_1\}$. Also there exists an ideal $I \in U_2 \cup U_3 \setminus \{\{0, a_2\}, \{0, a_3\}\}$. Put $V_1 := \{\{0, a_1\}, I_1, I_2\} \subseteq U_1$ and $V_2 := \{\{0, a_2\}, \{0, a_3\}, I\} \subseteq U_2 \cup U_3$. Hence we can find a copy of $K_{3,3}$ in the structure of $\mathbb{AG}(L)$, and so, by Kuratowski's Theorem, $\mathbb{AG}(L)$ is not planar.

Therefore, if $\mathbb{AG}(L)$ is planar, then $|\bigcup_{i=1}^3 U_i| \leq 6$ or at most for one of U_i 's, $|U_i| = 1$. Now, we have the following cases:

Case 1. $|\bigcup_{i=1}^3 U_i| = 3$. In this situation $\mathbb{AG}(L)$ is a unicycle graph as in Figure 2, and so it is planar.

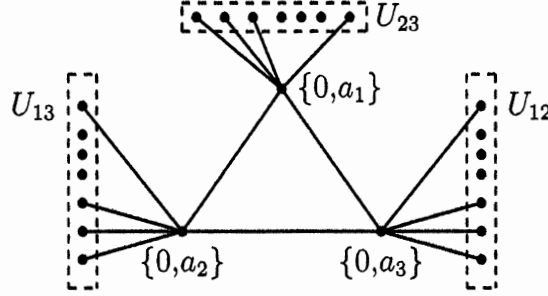


Figure 2:

Case 2. $|\bigcup_{i=1}^3 U_i| = 4$. Then there exists an ideal $I \in U_i \setminus \{0, a_i\}$. Therefore, $\mathbb{AG}(L)$ is isomorphic to the graph of Figure 3, and so $\mathbb{AG}(L)$ is planar.

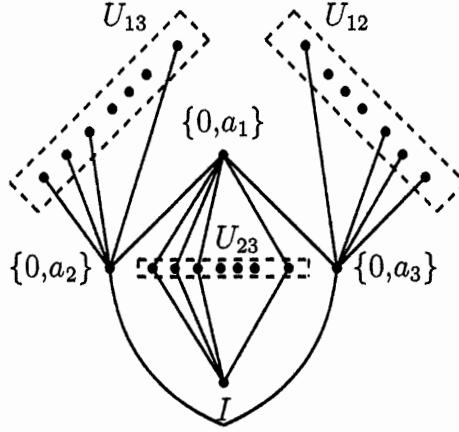


Figure 3:

Case 3. $|\bigcup_{i=1}^3 U_i| = 5$ and $|U_i| \leq 2$, for $i = 1, 2, 3$. Then there exist two distinct ideals I_1 and I_2 , such that $I_1 \in U_1 \setminus \{0, a_1\}$ and $I_2 \in U_2 \setminus \{0, a_2\}$.

If $|U_{j3}| \geq 1$, for all $1 \leq j \leq 2$, then there exist two ideals J_{13} and J_{23} in U_{j3} . So we can find a subdivision of K_5 in $\mathbb{AG}(L)$ as in Figure 4. Thus $\mathbb{AG}(L)$ is not planar.

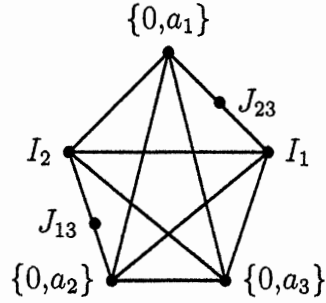


Figure 4:

If $U_{j3} = \emptyset$, for some $1 \leq j \leq 2$, then $\mathbb{AG}(L)$ is isomorphic to the graph in Figure 5, which is planar.

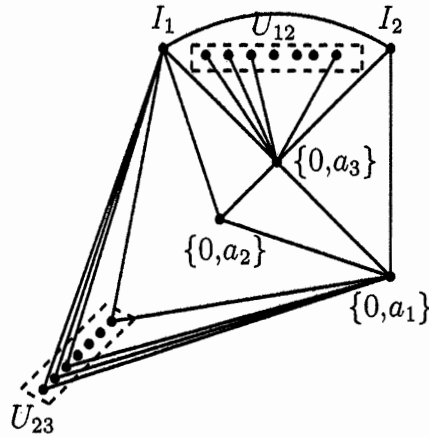


Figure 5:

Case 4. $|\bigcup_{i=1}^3 U_i| = 6$. We have the following situations:

(i) $|U_i| = 2$, for all $i = 1, 2, 3$. Set $U_i = \{\{0, a_i\}, I_i\}$ for all $i = 1, 2, 3$. Assume that at least one of the sets U_{ij} is non-empty, say U_{12} . Suppose

that $J \in U_{12}$. Then we can find a subdivision of $K_{3,3}$ in the graph $\mathbb{AG}(L)$, which is pictured in Figure 6.

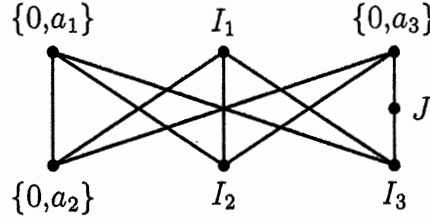


Figure 6:

If $U_{ij} = \emptyset$, for all $1 \leq i \neq j \leq 3$, then $\mathbb{AG}(L)$ is isomorphic to $K_{2,2,2}$, which is planar (see Figure 7).

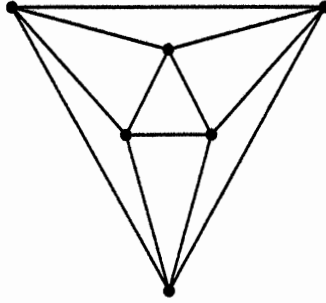


Figure 7:

(ii) $|U_i| = 3$, for some $i = 1, 2, 3$, say U_1 . Then there exist two distinct ideals $I_1, I_2 \in U_1 \setminus \{0, a_1\}$. Also there exists an ideal $I \in U_2 \cup U_3 \setminus \{\{0, a_2\}, \{0, a_3\}\}$. Put $V_1 := \{\{0, a_1\}, I_1, I_2\} \subseteq U_1$ and $V_2 := \{\{0, a_2\}, \{0, a_3\}, I\} \subseteq U_2 \cup U_3$. Clearly one can find a copy of $K_{3,3}$ in $\mathbb{AG}(L)$, and so, by Kuratowski's Theorem, $\mathbb{AG}(L)$ is not planar.

With the above discussion, we have the following theorem.

Theorem 2.4. *Suppose that $|A(L)| = 3$. Then $\mathbb{AG}(L)$ is planar if and only if one of the following conditions hold.*

- (i) $|\bigcup_{i=1}^3 U_i| \in \{3, 4\}$.

- (ii) $|\bigcup_{i=1}^3 U_i| = 5$ and $|U_i| \leq 2$, for all $1 \leq i \leq 3$.
- (iii) $|\bigcup_{i=1}^3 U_i| = 6$, $|U_i| = 2$, and $U_{ij} = \emptyset$, for all $1 \leq i \neq j \leq 3$.
- (iv) $|\bigcup_{i=1}^3 U_i| \geq 5$, $|U_i| = |U_j| = 1$ and $U_{ij} = \emptyset$, for some $1 \leq i \neq j \leq 3$.

Finally, in order to complete the study of the planarity of $\mathbb{AG}(L)$, we assume that $|A(L)| = 4$.

First suppose that $|\bigcup_{i=1}^4 U_i| \geq 6$. Then we have the following cases:

Case 1. $|U_i| \geq 3$, for some i with $1 \leq i \leq 4$. Without lose of generality, we may assume that $|U_1| \geq 3$. So there exist two distinct vertices $I, J \in U_1 \setminus \{0, a_1\}$. Therefore the vertices of the set $\{\{0, a_1\}, I, J\}$ are adjacent to the vertices of the set $\{\{0, a_2\}, \{0, a_3\}, \{0, a_4\}\}$. Thus we can find a copy of $K_{3,3}$ in $\mathbb{AG}(L)$, and hence $\mathbb{AG}(L)$ is not planar.

Case 2. For all $1 \leq i \leq 4$, $|U_i| \leq 2$. Without lose of generality, we may assume that there are vertices $I \in U_1 \setminus \{0, a_1\}$ and $J \in U_2 \setminus \{0, a_2\}$. Therefore the vertices of the set $\{\{0, a_1\}, \{0, a_3\}, I\}$ are adjacent to the vertices of the set $\{\{0, a_2\}, \{0, a_4\}, J\}$, and so $\mathbb{AG}(L)$ is not planar.

So if $\mathbb{AG}(L)$ is planar, then $|\bigcup_{i=1}^4 U_i| \leq 5$. Clearly, $|\bigcup_{i=1}^4 U_i| \geq 4$. So it is sufficient to investigate the cases that $|\bigcup_{i=1}^4 U_i| = 4$ or $|\bigcup_{i=1}^4 U_i| = 5$.

First, suppose that $|\bigcup_{i=1}^4 U_i| = 4$. We have the following situations:

(i) If $|U_{ij}| \geq 1$ and $|U_{tk}| \geq 1$, for some $1 \leq i \neq j \neq t \neq k \leq 4$, then there exist distinct vertices $I \in U_{ij}$ and $J \in U_{tk}$. Therefore the vertices of the sets $\{\{0, a_i\}, \{0, a_j\}, I\}$ and $\{\{0, a_t\}, \{0, a_k\}, J\}$ are adjacent, and so $\mathbb{AG}(L)$ is not planar.

(ii) If $U_{tk} = \emptyset$ whenever $U_{ij} \neq \emptyset$, for all $1 \leq i \neq j \neq t \neq k \leq 4$, then clearly $\mathbb{AG}(L)$ is planar.

Theorem 2.5. *Suppose that $|\bigcup_{i=1}^4 U_i| = 4$. Then $\mathbb{AG}(L)$ is planar if and only if $U_{tk} = \emptyset$ whenever $U_{ij} \neq \emptyset$, for all $1 \leq i \neq j \neq t \neq k \leq 4$.*

Proof. First suppose that $\mathbb{AG}(L)$ is planar. Then, by the above discussion, one can easily see that the result holds.

Conversely, assume that $U_{tk} = \emptyset$ whenever $U_{ij} \neq \emptyset$, for all $1 \leq i \neq j \neq t \neq k \leq 4$. Then $\mathbb{AG}(L)$ is isomorphic to the graph on Figure 8, which is planar. $\square$

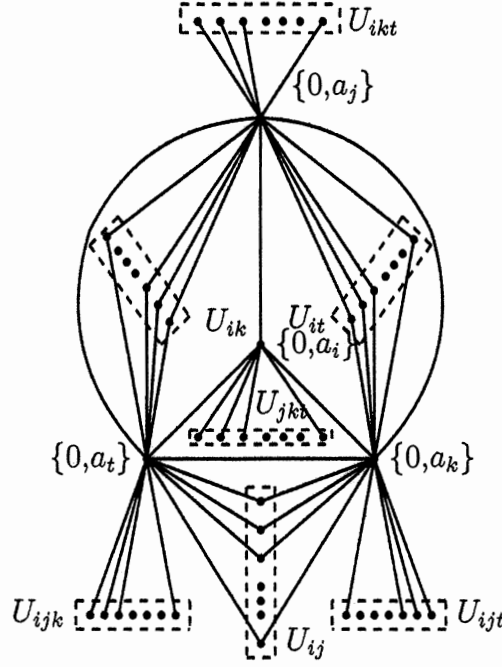


Figure 8:

Now, suppose that $|\bigcup_{i=1}^4 U_i| = 5$. Clearly, we have that $|U_i| = 2$, for some $1 \leq i \leq 4$. Without loss of generality, we may assume that $U_1 = \{\{0, a_1\}, I\}$.

Now, we have the following theorem.

Theorem 2.6. *Let $|\bigcup_{i=1}^4 U_i| = 5$ and $|U_1| = 2$. Then $\mathbb{AG}(L)$ is planar if and only if $U_{ij} = \emptyset$, for all $2 \leq i \neq j \leq 4$, and $U_{234} = \emptyset$.*

Proof. First assume that $\mathbb{AG}(L)$ is planar and suppose on the contrary that $U_{ij} \neq \emptyset$, for some $2 \leq i \neq j \leq 4$ or $U_{234} \neq \emptyset$. Then we have the following cases:

Case 1. $U_{ij} \neq \emptyset$, for some $2 \leq i \neq j \leq 4$. Then there exists an ideal $J \in U_{ij}$. Therefore, the vertices of the set $\{\{0, a_1\}, \{0, a_t\}, I\}$ are adjacent to the vertices of the set $\{\{0, a_i\}, \{0, a_j\}, J\}$, for all $2 \leq i \neq j \neq t \leq 4$. So $K_{3,3}$ is isomorphic to a subgraph of $\mathbb{AG}(L)$, and hence $\mathbb{AG}(L)$ is not planar, a contradiction.

Case 2. $U_{234} \neq \emptyset$. Then there exists an ideal k in U_{234} . So one can find a subdivision of K_5 in $\mathbb{AG}(L)$ as in Figure 9. Thus $\mathbb{AG}(L)$ is not planar.

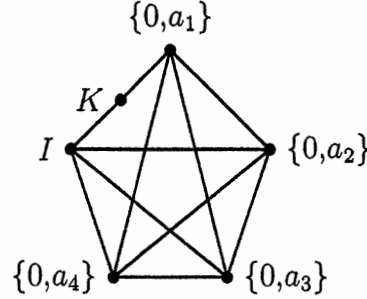


Figure 9:

So if $\mathbb{AG}(L)$ is planar, then $U_{ij} = \emptyset$, for all $2 \leq i \neq j \leq 4$, and $U_{234} = \emptyset$.

For the converse statement, $\mathbb{AG}(L)$ is pictured in Figure 10, which is planar. $\square$

3 Outerplanarity of $\mathbb{AG}(L)$

A directed graph is *outerplanar* if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

In the rest of the paper, we characterize all lattices L , such that $\mathbb{AG}(L)$ is outerplanar.

Lemma 3.1. *If $\mathbb{AG}(L)$ is outerplanar, then $|A(L)| \leq 3$.*

Proof. Assume to the contrary that $|A(L)| \geq 4$. Since the induced subgraph of $\mathbb{AG}(L)$ on vertex-set $\{\{0, a_i\}\}$, for $1 \leq i \leq 4$, is a complete subgraph, we can find a copy of K_4 in $\mathbb{AG}(L)$, and so $\mathbb{AG}(L)$ is not outerplanar. Hence we have $|A(L)| \leq 3$. $\square$

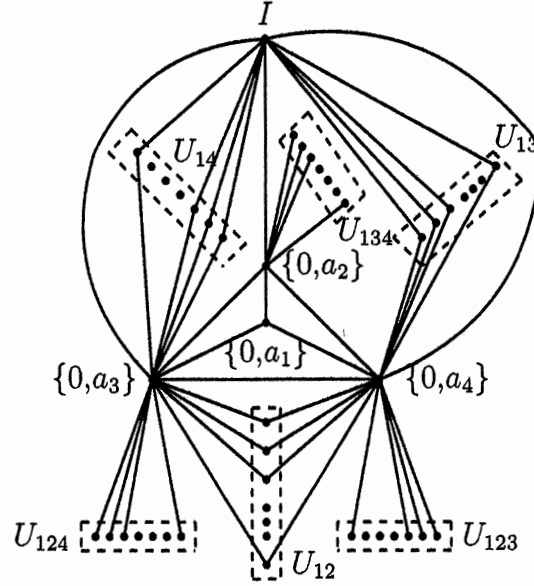


Figure 10:

By Lemma 3.1, we study the cases that $|A(L)|$ is equal to 2 or 3. In the following proposition, we investigate the outerplanarity of $\mathbb{AG}(L)$, when $|A(L)| = 2$.

Proposition 3.2. *Assume that $|A(L)| = 2$. Then $\mathbb{AG}(L)$ is outerplanar if and only if one of the following assertions hold.*

- (i) $|U_i| = 1$, for some $1 \leq i \leq 2$.
- (ii) $|U_i| \leq 2$, for all $1 \leq i \leq 2$.

Proof. Since $|A(L)| = 2$, we have that $\mathbb{AG}(L)$ is a complete bipartite graph. Now one can easily see that $\mathbb{AG}(L)$ is outerplanar if and only if $|U_i| = 1$, for some $1 \leq i \leq 2$, or $|U_i| \leq 2$, for all $1 \leq i \leq 2$. $\square$

In the following, we check the outerplanarity of $\mathbb{AG}(L)$, when $|A(L)| = 3$.

If $|\bigcup_{i=1}^3 U_i| \geq 5$, then we can find a copy of $K_{2,3}$ in the structure of $\mathbb{AG}(L)$, and so $\mathbb{AG}(L)$ is not outerplanar. Therefore, if $\mathbb{AG}(L)$ is outerplanar, then $|\bigcup_{i=1}^3 U_i| \leq 4$. Now, we have the following cases:

Case 1. $|\bigcup_{i=1}^3 U_i| = 3$. In this situation $\mathbb{AG}(L)$ is a unicyclic graph which is pictured in Figure 11, and so it is outerplanar.

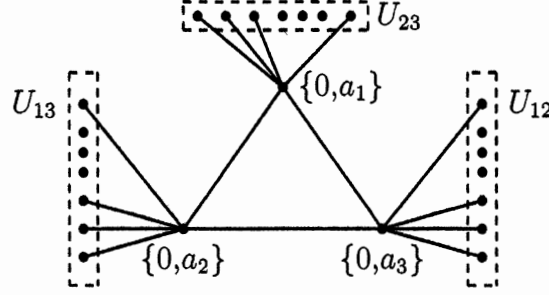


Figure 11:

Case 2. $|\bigcup_{i=1}^3 U_i| = 4$. Then we may assume that $|U_i| = 2$, for some $1 \leq i \leq 3$. So $U_i = \{\{0, a_i\}, I\}$, for some $1 \leq i \leq 3$. Now we have the following situations:

(i) If $|U_{jk}| \geq 1$, for some $1 \leq i \neq j \neq k \leq 3$, then we can find a copy of $K_{2,3}$ with vertex-set $\{\{0, a_1\}, I\} \cup \{\{0, a_2\}, \{0, a_3\}, J\}$, where $J \in U_{jk}$, for some $1 \leq i \neq j \neq k \leq 3$, and so $\mathbb{AG}(L)$ is not outerplanar.

(ii) If $U_{jk} = \emptyset$, for all $1 \leq i \neq j \neq k \leq 3$, then $\mathbb{AG}(L)$ is isomorphic to the graph which is pictured in Figure 12, and so $\mathbb{AG}(L)$ is outerplanar.

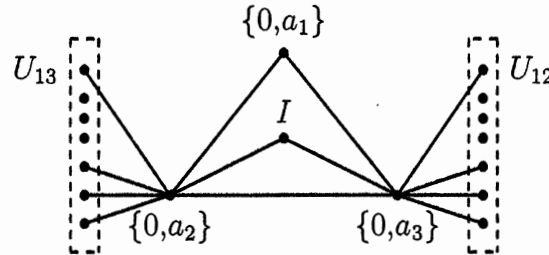


Figure 12:

By the above discussion we have the following theorem.

Theorem 3.3. Suppose that $|A(L)| = 3$. Then $\mathbb{AG}(L)$ is outerplanar if and only if one of the following conditions hold.

- (i) $|\bigcup_{i=1}^3 U_i| = 3$.
- (ii) $|\bigcup_{i=1}^3 U_i| = 4$ and if $|U_i| = 2$ for some $1 \leq i \leq 3$, then $U_{jk} = \emptyset$, for all $1 \leq i \neq j \neq k \leq 3$.

Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G . Let C be a cycle of G . We say that C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive cycle property (PCP)* if any two primitive cycles intersect in at most one edge. The number $\text{frank}(G)$ is called the *free rank* of G and it is the number of primitive cycles of G . Also, the number $\text{rank}(G) = q - n + r$ is called the *cycle rank* of G , where r is the number of connected components of G . The cycle rank of G can be expressed as the dimension of the cycle space of G . By [11, Proposition 2.2], we have $\text{rank}(G) \leq \text{frank}(G)$. A graph G is called a *ring graph* if it satisfies in one of the following equivalent conditions (see [11]).

- (i) $\text{rank}(G) = \text{frank}(G)$,
- (ii) G satisfies the *PCP* and G does not contain a subdivision of K_4 as a subgraph.

Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

Now, in view of the proofs of Proposition 3.2 and Theorem 3.3, we have the following result.

Theorem 3.4. *The annihilating-ideal graph $\mathbb{AG}(L)$ is a ring graph if and only if it is an outerplanar graph.*

Acknowledgments. The author would like to express her deep gratitude to Prof. Kazem Khashyarmanesh and Dr. Mojgan Afkhami for many valuable comments which have greatly improved the presentation of the paper.

References

- [1] M. AFKHAMİ, S. BAHRAMI, K. KHASHYARMANESH, F. SHAHSAVAR, *The annihilating-ideal graph of a lattice*, Georgian Math. J. (to appear).

- [2] M. AFKHAMI, K. KHASHYARMANESH, *The cozero-divisor graph of a commutative ring*, Southeast Asian Bull. Math. **35** (2011) 753-762.
- [3] D.F. ANDERSON, M.C. AXTELL, J.A. STICKLES, *Zero-divisor graphs in commutative rings*, *Commutative Algebra, Noetherian and Non-Noetherian Perspectives* (M. Fontana, S.E. Kabbaj, B. Olberding, I. Swanson) Springer-Verlag, New York, 2011, 23-45.
- [4] D.F. ANDERSON, P.S. LIVINGSTON, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999) 434-447.
- [5] I. BECK, *Coloring of commutative rings*, J. Algebra **116** (1998) 208-226.
- [6] M. BEHBOODI, Z. RAKEEI, *The annihilating-ideal graph of commutative rings I*, J. Algebra Appl. **10** (2011) 727-739.
- [7] M. BEHBOODI, Z. RAKEEI, *The annihilating-ideal graph of commutative rings II*, J. Algebra Appl. **10** (2011) 741-753.
- [8] J.A. BONDY, U.S.R. MURTY, *Graph Theory with Applications*, American Elsevier, New York, 1976.
- [9] B.A. DAVEY, H.A. PRIESTLEY, *Introduction to Lattices and Order*, Cambridge University Press, 2002.
- [10] E. ESTAJI, K. KHASHYARMANESH, *The zero-divisor graph of a lattice*, Results. Math. **61** (2012) 1-11.
- [11] I. GITLER, E. REYES, R.H. VILLARREAL, *Ring graphs and complete intersection toric ideals*, Discrete Math. **310** (2010) 430-441.
- [12] A.V. KELAREV, *Graph Algebras and Automata*, Marcel Dekker, New York, 2003.
- [13] A.V. KELAREV, J. RYAN, J. YEARWOOD, *Cayley graphs as classifiers for data mining: the influence of asymmetries*, Discrete Math. **309** (2009) 5360-5369.

- [14] J.B. NATION, *Notes on lattice theory*, Cambridge studies in advanced mathematics, Vol. 60, Cambridge university press, Cambridge, 1998.

**MATHEMATICAL MODELING OF SPATIAL AND TEMPORAL
NONLOCALITIES ON NONLINEAR WAVES EVOLUTION¹**

**Danylenko V.A., Danevich T.B., Makarenko O.S.²,
Skurativsky S.I. and Vladimirov V.A.**

Subbotin Institute of Geophysics of the National Academy of Sciences of Ukraine
Palladin Av. 32
Kyiv, Ukraine, UA-03680
makalex@i.com.ua

Received December 22, 2014

Abstract

In this paper we discuss the results of the investigation of nonlinear waves in media with different nonlocal dynamical equations of state. In particular, we deal with the propagation of the small disturbance spreading as a travelling wave slowly varying. Using the perturbation theory, the amplitude equations belonging to nonintegrable equations were derived. Applying the method of singular manifolds described by Kudryashov the exact analytical solutions of the amplitude equations were constructed.

¹Based on invited talks given by A. Makarenko at the 6th Petrov International Symposium on *High Energy Physics, Cosmology and Gravity*, Kyiv–Kosivska Poliana (Ukraine), September 5–20, 2013.

²On leave from the National Technical University of Ukraine (KPI), Ukraine.

1. The evolution of nonlinear waves within the framework of the nonlocal dynamical equation of state with the first order time and the second order spatial derivatives

Consider the propagation of the small disturbance spreading as a travelling wave slowly varying in nonlocal nonlinear media. We choose the dynamical equation of state for media written in the Euler coordinates in the form of relation:

$$\begin{aligned} & \frac{1}{\rho^2} \left\{ \frac{\Gamma \varepsilon_r}{\tau_{TP}} \left[-\rho_{xx} + \frac{1}{\rho} (\rho_x)^2 \right] + \frac{1}{\tau_{TP}} \frac{d\rho}{dt} \right\} + \omega_0^2 \sigma(S) \rho_0^{1-\Gamma_{V0}} \rho^{\Gamma_{V0}} - \omega_0^2 \rho_0 = \\ & = b(p - p_0) + b\tau_{TV} \frac{dp}{dt} - \frac{\alpha_\infty^2}{\rho_0 \alpha_0} \frac{1}{\tau_{PV}} \frac{dT}{dt} - b\Gamma \varepsilon_r \tau_{TV} \left(p_{xx} + \frac{\rho_x}{\rho} p_x \right) + \\ & + \frac{\alpha_\infty^2}{\rho_0 \alpha_0} \frac{\Gamma \varepsilon_r}{\tau_{PV}} \left(T_{xx} + \frac{\rho_x}{\rho} T_x \right). \end{aligned} \quad (1)$$

with omitting the temperature terms (a similar model has been used in papers [1, 2])

$$\begin{aligned} & \rho_t = -(\rho v)_x, \quad \rho v_t + \rho v v_x = -p_x, \\ & \frac{1}{\rho^2} \left\{ \frac{\Gamma \varepsilon_r}{\tau_{TV}} \left[-\rho_{xx} + \frac{1}{\rho} (\rho_x)^2 \right] + \frac{1}{\tau_{TP}} \frac{d\rho}{dt} \right\} + \omega_0^2 \rho_0^{1-\Gamma_{V0}} \rho^{\Gamma_{V0}} - \omega_0^2 \rho_0 = \\ & = b(p - p_0) + b\tau_{TV} \frac{dp}{dt} - b\Gamma \varepsilon_r \tau_{TV} [p_{xx} + \rho_x \rho^{-1} p_x], \end{aligned} \quad (2)$$

where

$$b \equiv v_0 \chi_{T\infty} / \tau_{TV} \tau_{TP}, \quad \omega_0^2 \equiv b c_{S0}^2 \alpha_0 T_0 / \gamma_0. \quad (3)$$

Using the ideas of the perturbation theory, we are going to find the solutions of this system as small perturbations of equilibrium state $(\partial_0, \rho_0, 0)$

$$\begin{aligned} p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + O(\varepsilon^3), \\ \rho &= \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + O(\varepsilon^3), \\ v &= \varepsilon v_1 + \varepsilon^2 v_2 + O(\varepsilon^3), \end{aligned} \quad (4)$$

where ε is the small parameter.

Similar to papers [28, 29], we consider the travelling wave slowly varying. Let us introduce the new variables

$$\varsigma = \varepsilon^\alpha (x - ct), \tau = a\varepsilon^\beta t, \quad (5)$$

where c is the travelling wave velocity. System (2) in variables (ς, τ) takes the form

$$\begin{aligned} a\varepsilon^{(\beta-\alpha)} \frac{\partial \rho}{\partial \tau} + (v - c) \frac{\partial \rho}{\partial \varsigma} &= -\rho \frac{\partial v}{\partial \varsigma}, \\ a\varepsilon^{(\beta-\alpha)} \frac{\partial v}{\partial \tau} + (v - c) \frac{\partial v}{\partial \varsigma} &= -\frac{1}{\rho} \frac{\partial p}{\partial \varsigma}, \\ \frac{1}{\rho^2} \left\{ \frac{\Gamma \varepsilon_r}{\tau_{TP}} \left[-\varepsilon^{2\alpha} \frac{\partial^2 \rho}{\partial \varsigma^2} + \frac{1}{\rho} \varepsilon^{2\alpha} \left(\frac{\partial \rho}{\partial \varsigma} \right)^2 \right] + \frac{\varepsilon^\alpha}{\tau_{TP}} \left[a\varepsilon^{(\beta-\alpha)} \frac{\partial \rho}{\partial \tau} + (v - c) \frac{\partial \rho}{\partial \varsigma} \right] \right\} + \\ \omega_0^2 \rho_0^{(1-\Gamma_{V0})} \rho^{\Gamma_{V0}} - \omega_0^2 \rho_0 &= b(p - p_0) + b\tau_{TV} \varepsilon^\alpha \left[a\varepsilon^{(\beta-\alpha)} \frac{\partial p}{\partial \tau} + (v - c) \frac{\partial p}{\partial \varsigma} \right] - \\ -b\Gamma \varepsilon_x \tau_{TV} \left[\varepsilon^{2\alpha} \frac{\partial^2 p}{\partial \varsigma^2} + \varepsilon^{2\alpha} \frac{\partial p}{\partial \varsigma} \frac{1}{\rho} \frac{\partial \rho}{\partial \varsigma} \right], \end{aligned} \quad (6)$$

where the derivatives in variables (ς, τ) and (x, t) are connected by means of the relations

$$\begin{aligned} \frac{\partial}{\partial t} &= a\varepsilon^\beta \frac{\partial}{\partial \tau} - \varepsilon^\alpha \frac{\partial}{\partial \varsigma}, \quad \frac{d}{dt} = \varepsilon^\alpha \left[a\varepsilon^{(\beta-\alpha)} \frac{\partial}{\partial \tau} + (v - c) \frac{\partial}{\partial \varsigma} \right], \\ \frac{\partial}{\partial x} &= \varepsilon^\alpha \frac{\partial}{\partial \varsigma}, \quad \frac{\partial^2}{\partial x^2} = \varepsilon^{2\alpha} \frac{\partial^2}{\partial \varsigma^2}. \end{aligned} \quad (7)$$

Let $\beta = \alpha + 1$. Inserting (5) into (6) in the first approximation we get

$$\rho_1 = \frac{\rho_0}{c} v_1, \quad p_1 = c\rho_0 v_1, \quad p_1 = \frac{c_{s0}^2 (\gamma_0 - 1)}{\gamma_0} \rho_1, \quad (8)$$

Analyzing the conditions of compatibility the travelling wave velocity is defined

$$c^2 = \frac{c_{s0}^2 (\gamma_0 - 1)}{\gamma_0}. \quad (9)$$

Thus, from the first approximation of the perturbation theory, it follows that the slowly varying travelling wave in model media propagates with the velocity, which is proportional to the equilibrium velocity of sound.

In the second approximation of the perturbation theory, we derive the following equations

$$\frac{\partial p_2}{\partial \varsigma} = \left[-\rho_0 a \frac{\partial v_1}{\partial \tau} + \rho_0 c \frac{\partial v_2}{\partial \varsigma} \right], \quad (10)$$

$$\frac{\partial \rho_2}{\partial \varsigma} = \left[\frac{a \rho_0}{c} \frac{\partial v_1}{\partial \tau} + \frac{2 \rho_0 v_1}{c} \frac{\partial v_1}{\partial \varsigma} + \rho_0 \frac{\partial v_2}{\partial \varsigma} \right], \quad (11)$$

$$\frac{\partial v_1}{\partial \tau} + v_1 \frac{\partial v_1}{\partial \varsigma} + A \varepsilon^{(\alpha-1)} \frac{\partial^2 v_1}{\partial \varsigma^2} + \frac{\Gamma \varepsilon_r}{c} A \varepsilon^{(2\alpha-1)} \frac{\partial^3 v_1}{\partial \varsigma^3} = 0, \quad (12)$$

where $a = \frac{(\Gamma_{V0} + 1)}{2}$, $A = \frac{1}{b(\Gamma_{V0} + 1) \tau_{TP}} (b \tau_{TP} \tau_{TV} c^2 - v_0^2)$

The application of expressions (3), (9) for b , c^2 and relation

$$\chi_{T\infty} = v_0 c_{T\infty}^{-2} = \gamma_\infty v_0 c_{S\infty}^{-2}, \quad (13)$$

yields

$$A = \frac{\tau_{TV}}{(\Gamma_{V0} + 1)} \left(\frac{c_{S0}^2 (\gamma_0 - 1)}{\gamma_0} - \frac{c_{S\infty}^2}{\gamma_\infty} \right). \quad (14)$$

Let $\alpha = 1$, ε_r be a finite number or $\varepsilon_r \rightarrow 0$. Then equation (12) contains small parameter with the leading derivative

$$\frac{\partial v_1}{\partial \tau} + v_1 \frac{\partial v_1}{\partial \varsigma} + A \frac{\partial^2 v_1}{\partial \varsigma^2} + \frac{\Gamma \varepsilon_r}{c} A \varepsilon \frac{\partial^3 v_1}{\partial \varsigma^3} = 0 \quad (15)$$

Let $\alpha = 1/2$, ε_r be a finite number. Evolution equation (12) is reduced to the KdVB equation

$$\frac{\partial v_1}{\partial \tau} + v_1 \frac{\partial v_1}{\partial \varsigma} + \frac{\Gamma \varepsilon_r}{c} A \frac{\partial^3 v_1}{\partial \varsigma^3} + A \frac{1}{\sqrt{\varepsilon}} \frac{\partial^2 v_1}{\partial \varsigma^2} = 0. \quad (16)$$

Let us rewrite equation (16) in the form

$$(v_1)_\tau + v_1 (v_1)_{\varsigma\varsigma\varsigma} + m (v_1)_\varsigma = n (v_1)_{\varsigma\varsigma}, \quad m = \frac{\Gamma \varepsilon_r}{c} A, \quad n = -\frac{A}{\sqrt{\varepsilon}}. \quad (17)$$

Using the Backlund transformation analogous to paper [3], we find the exact solution to equation (17)

$$v_1(\varsigma, \tau) = 12m \frac{\partial^2}{\partial \varsigma^2} \ln F - \frac{12n}{5} \frac{\partial}{\partial \varsigma} \ln F + v_2, v_2 = \text{const} = C_1.$$

The function

$$F(\varsigma, \tau) = C_2 + C_3 \exp(k\varsigma + \omega\tau),$$

$$\omega = -C_1 k + \frac{6nk^2}{5}, \quad k = \pm n/5m$$

leads to the exact solutions of the KdVB equation (16):

$$v_1(\varsigma, \tau) = C_1 + \frac{3n^2}{25m} (1 - U^2) - \frac{6nk}{5} (1 + U), \quad U = th \left\{ \frac{1}{2} (k\varsigma + \omega\tau) \right\}.$$

In our case

$$v_1(\varsigma, \tau) = C_1 + \frac{3Ac}{25\epsilon\Gamma\epsilon_r} (1 - U^2) + \frac{6Ak}{5\sqrt{\epsilon}} (1 - U).$$

The amplitude of this wave is constant and is equal to

$$\frac{12n^2}{25m} = \frac{12Ac}{25\epsilon\Gamma\epsilon_r},$$

The velocity equals

$$C_0 = -\frac{\omega}{k} = C_1 - \frac{6nk}{5} = C_1 + \frac{6Ak}{5\sqrt{\epsilon}}.$$

Let $\alpha = 1/2$, but if $A \rightarrow 0$ faster as $\epsilon \rightarrow 0$, and ϵ_r tends to ∞ such that $\Gamma\epsilon_r A \neq 0$, then from evolution equation (12) it follows

$$\frac{\partial v_1}{\partial \tau} + v_1 \frac{\partial v_1}{\partial \varsigma} + D \frac{\partial^3 v_1}{\partial \varsigma^3} = 0,$$

$$D \equiv \frac{\Gamma\epsilon_r A}{c} = \frac{\Gamma\epsilon_r}{c} \frac{1}{b(\Gamma v_0 + 1) \tau_{TP}} (b\tau_{TP}\tau_{TV}c^2 - v_0^2), \quad (18)$$

Note that the stationary solution of equation (18) tends to the soliton solution. If we change the variable $\xi = \varsigma - c'\tau$, then after integrating we get

$$-c'v_1 + \frac{1}{2}(v_1)^2 + D\frac{d^2v_1}{d^2\xi} = 0. \quad (19)$$

We look for the solution of equation (19) in the form

$$v_1 = v_{10} \cosh^{-2} \left(\frac{\xi}{h} \right), \quad (20)$$

where v_{10} is the amplitude, h is the width of the soliton. Using the properties of the hyperbolic functions

$$\begin{aligned} \xi \rightarrow 0, \quad \cosh(\xi) &= 1, \quad \sinh(\xi) = 0; \\ \xi \rightarrow \pm\infty, \quad \cosh(\xi) &= \sinh(\xi) \rightarrow \exp(\xi)/2 \end{aligned}$$

and inserting (20) into (19), we obtain two algebraic equations with respect to width h and amplitude v_{10} :

$$h = 2\sqrt{D/c'}, \quad v_{10} = 3c',$$

where c' is the soliton velocity, parameter D is defined in (18).

Thus the following solution of equation (19) is found

$$v_1 = 3c' \cosh^{-2} \left\{ \frac{\xi}{2} \sqrt{\frac{b(\Gamma_{v0} + 1) c' \tau_{TP}}{\Gamma \varepsilon_r (b \tau_{TP} \tau_{TV} c^2 - v_0^2)}} \right\}.$$

2. Evolution of nonlinear waves: the complete temporally nonlocal dynamical equation of state with the first order time derivatives and the linear dynamical part

Consider the dynamical equation of state:

$$p - p_0 = \frac{c_{S0}^2}{\gamma_0 v_0} \left\{ \frac{(1 - \gamma_0)}{3\Gamma_{v0}} \left[\left(\frac{v}{v_0} \right)^{-1} - 1 \right] + \frac{(\gamma_0 - 1)}{\Gamma_{v0}} \left[\sigma(S) \left(\frac{v}{v_0} \right)^{-(\Gamma_{v0}+1)} - 1 \right] \right\} - \tau_{TV} \frac{dp}{dt} - \frac{\tau_{TP}}{v_0 \chi_{T0}} \frac{dv}{dt} + \frac{\alpha_0}{\chi_{T0}} \tau_{PV} \frac{dT}{dt}. \quad (21)$$

in which the quasistationary part is equation

$$p - p_0 = \frac{c_{S0}^2}{\gamma_0 v_0} \left\{ \frac{(1 - \gamma_0)}{3\Gamma_{v0}} \left[\left(\frac{v}{v_0} \right)^{-1} - 1 \right] + \frac{(\gamma_0 - 1)}{\Gamma_{v0}} \left[\sigma(S) \left(\frac{v}{v_0} \right)^{-(\Gamma_{v0}+1)} - 1 \right] \right\}, \quad (22)$$

and the dynamical part coincides with formula

$$-\tau_{TV} \frac{dp}{dt} - \frac{\tau_{TP}}{v_0 \chi_{T0}} \frac{dv}{dt} + \frac{\alpha_0}{\chi_{T0}} \tau_{PV} \frac{dT}{dt}. \quad (23)$$

Equation (21) is the complete temporally nonlocal dynamical equation of state for media with linear dynamical part. Excluding the expression dT/dt from equation (21) by means of formula (??), multiplying the result by $bv^{(\Gamma_{v0}+1)}$, and using the formulae

$$\frac{c_{S0}^2 (\gamma_0 - 1) b}{\gamma_0 \Gamma_{v0}} \equiv \omega_0^2, \quad \frac{b}{v_0 \chi_{T0}} = \frac{1}{\tau_{TP}^2},$$

$$\tau_{TP} = \tau_{TV} \frac{\chi_{T0}}{\chi_{T\infty}} = \tau_{PV} \frac{\alpha_0}{\alpha_\infty} \Rightarrow \alpha_\infty = \frac{\tau_{PV}}{\tau_{TP}} \alpha_0.$$

we obtain

$$-\frac{1}{\tau_{TP}} \left[v^{(\Gamma_{v0}+1)} + \alpha_\infty T_0 \Gamma_{v0} \sigma(S) v_0^{(\Gamma_{v0}+1)} \right] \frac{dv}{dt} + \omega_0^2 v_0^{\Gamma_{v0}} \sigma(S) - \frac{2}{3} \omega_0^2 v_0^{-1} v^{(\Gamma_{v0}+1)} - \frac{\omega_0^2}{3} v^{\Gamma_{v0}} = b v^{(\Gamma_{v0}+1)} (p - p_0) + b \tau_{TV} v^{(\Gamma_{v0}+1)} \frac{dp}{dt}. \quad (24)$$

Consider the evolution of travelling waves slowly varying in media with equation of state (24). In the Lagrange coordinates (r, t) the initial system has the form

$$\frac{\partial v}{\partial t} - v_0 \frac{\partial u}{\partial r} = 0, \quad \frac{1}{v_0} \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0, \quad (25)$$

$$\begin{aligned} & -\frac{1}{\tau_{TP}} \left[v^{(\Gamma_{V0}+1)} + \alpha_\infty T_0 \Gamma_{V0} \sigma(S) v_0^{(\Gamma_{V0}+1)} \right] \frac{\partial v}{\partial t} + \omega_0^2 v_0^{\Gamma_{V0}} \sigma(S) - \\ & -\frac{2}{3} \omega_0^2 v_0^{-1} v^{(\Gamma_{V0}+1)} - \frac{\omega_0^2}{3} v^{\Gamma_{V0}} = b v^{(\Gamma_{V0}+1)} (p - p_0) + b \tau_{TV} v^{(\Gamma_{V0}+1)} \frac{\partial p}{\partial t}. \end{aligned} \quad (26)$$

Now we use the travelling coordinates

$$\varsigma = \varepsilon^\alpha (r - ct); \theta = a \varepsilon^{(\alpha+1)} t \quad (27)$$

and expansions in series with respect to the small parameter

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots, \quad v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots, \quad u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots \quad (28)$$

Substituting (27) and (28) into system (26) with the expression

$$(v_0 + \varepsilon v_1 + \varepsilon^2 v_2)^n \approx v_0^n + n v_0^{(n-1)} \varepsilon v_1 + n v_0^{(n-1)} \varepsilon^2 v_2 + \frac{(n-1)n}{2} v_0^{[n-2]} \varepsilon^2 v_1^2 \quad (29)$$

in the first order approximation we obtain

$$v_1 = -\frac{v_0}{c} u_1, \quad p_1 = \frac{c}{v_0} u_1 \quad (30)$$

and

$$-\omega_0^2 v_1 \left[\frac{2}{3} (\Gamma_{V0} + 1) + \frac{1}{3} \Gamma_{V0} \right] v_0^{(\Gamma_{V0}-1)} = b v_0^{(\Gamma_{V0}+1)} p_1 \Rightarrow -\omega_0^2 v_1 \left(\Gamma_{V0} + \frac{2}{3} \right) = b v_0^2 p_1.$$

Since $\frac{\omega_0^2}{b} = \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0}$, then

$$\left(\Gamma_{V0} + \frac{2}{3} \right) \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0} = -\frac{v_0^2 p_1}{v_1}. \quad (31)$$

By inserting (30) into (31), we find the expression defining the velocity of the small disturbances in media with dynamical equation of state (24)

$$c^2 = \left(\Gamma_{V0} + \frac{2}{3} \right) \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0} = \frac{c_{S0}^2 (\gamma_0 - 1)}{\gamma_0} + \frac{2}{3} \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0}. \quad (32)$$

In the second order approximation we get [4, 5]:

$$\frac{\partial v_2}{\partial \varsigma} = \frac{a}{c} \frac{\partial v_1}{\partial \theta} - \frac{v_0}{c} \frac{\partial u_2}{\partial \varsigma}, \quad \frac{\partial p_2}{\partial \varsigma} = -\frac{a}{v_0} \frac{\partial u_1}{\partial \theta} + \frac{c}{v_0} \frac{\partial u_2}{\partial \varsigma}. \quad (33)$$

and

$$\begin{aligned} & \frac{\varepsilon^{(\alpha-1)}}{\tau_{TP}} v_0^{(\Gamma_{V0}+1)} [1 + \alpha_\infty T_0 \Gamma_{V0} \sigma(S)] c \frac{\partial v_1}{\partial \varsigma} - \\ & - \frac{2}{3} \omega_0^2 v_0^{-1} (\Gamma_{V0} + 1) v_0^{\Gamma_{V0}} v_2 - \frac{2}{3} \omega_0^2 v_0^{-1} \frac{\Gamma_{V0} (\Gamma_{V0} + 1)}{2} v_0^{(\Gamma_{V0}-1)} v_1^2 - \\ & - \frac{\omega_0^2}{3} \Gamma_{V0} v_0^{(\Gamma_{V0}-1)} v_2 - \frac{\omega_0^2 \Gamma_{V0} (\Gamma_{V0} - 1)}{3 \cdot 2} v_0^{(\Gamma_{V0}-1)} v_1^2 = \\ & = b (\Gamma_{V0} + 1) v_0^{\Gamma_{V0}} v_1 p_1 + b v_0^{(\Gamma_{V0}+1)} p_2 - b \tau_{TV} v_0^{(\Gamma_{V0}+1)} c \varepsilon^{(\alpha-1)} \frac{\partial p_1}{\partial \varsigma}. \end{aligned} \quad (34)$$

Let us divide equation (34) into $v_0^{(\Gamma_{V0}+1)}$, then differentiate with respect to ξ and insert expressions (30), (33). Then we obtain

$$\begin{aligned} & -\frac{1}{\tau_{TP}} [1 + \alpha_\infty T_0 \Gamma_{V0} \sigma(S)] \varepsilon^{(\alpha-1)} v_0 \frac{\partial^2 u_1}{\partial \varsigma^2} - \frac{\omega_0^2}{v_0^2} \left(\Gamma_{V0} + \frac{2}{3} \right) \left[-\frac{a v_0}{c^2} \frac{\partial u_1}{\partial \theta} - \frac{v_0}{c} \frac{\partial u_2}{\partial \varsigma} \right] - \\ & - \frac{\omega_0^2}{v_0 c^2} \Gamma_{V0} \left(\Gamma_{V0} + \frac{1}{3} \right) u_1 \frac{\partial u_1}{\partial \varsigma} = -2 \frac{b (\Gamma_{V0} + 1)}{v_0} u_1 \frac{\partial u_1}{\partial \varsigma} + \\ & + b \left[-\frac{a}{v_0} \frac{\partial u_1}{\partial \theta} + \frac{c}{v_0} \frac{\partial u_2}{\partial \varsigma} \right] - b \tau_{TV} \frac{c^2}{v_0} \varepsilon^{(\alpha-1)} \frac{\partial^2 u_1}{\partial \varsigma^2}. \end{aligned} \quad (35)$$

Taking into account

$$\begin{aligned} \frac{\omega_0^2}{v_0 c^2} \left(\Gamma_{V0} + \frac{2}{3} \right) a \frac{\partial u_1}{\partial \theta} &= \frac{ab}{v_0} \frac{\partial u_1}{\partial \theta}; \quad \frac{\omega_0^2}{v_0 c} \left(\Gamma_{V0} + \frac{2}{3} \right) \frac{\partial u_2}{\partial \xi} = \frac{bc}{v_0} \frac{\partial u_2}{\partial \xi}; \\ \frac{\omega_0^2}{v_0 c^2} \Gamma_{V0} \left(\Gamma_{V0} + \frac{1}{3} \right) &= \frac{b}{v_0} \frac{\Gamma_{V0} (3\Gamma_{V0} + 1)}{(3\Gamma_{V0} + 2)}; \end{aligned}$$

$$-\frac{(v_0 + \alpha_\infty v_0 T_0 \Gamma_{v0} \sigma(S))}{\tau_{TP}} + b\tau_{TV} \frac{c^2}{v_0} = \frac{1}{\tau_{TP}} \left[\frac{b\tau_{TV} \tau_{TP} c^2}{v_0} - (1 + \alpha_\infty T_0 \Gamma_{v0} \sigma(S)) \right],$$

equation (35) can be written in the form :

$$\begin{aligned} 2a \frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \varsigma} \left[-\frac{\Gamma_{v0} (3\Gamma_{v0} + 1)}{(3\Gamma_{v0} + 2)} + 2(\Gamma_{v0} + 1) \right] + \\ + \frac{1}{b\tau_{TP}} [b\tau_{TV} \tau_{TP} c^2 - (1 + \alpha_\infty T_0 \Gamma_{v0} \sigma(S)) v_0^2] \varepsilon^{(\alpha-1)} \frac{\partial^2 u_1}{\partial \varsigma^2} = 0. \end{aligned} \quad (36)$$

Denote

$$a \equiv \frac{1}{2} \left[-\frac{\Gamma_{v0} (3\Gamma_{v0} + 1)}{(3\Gamma_{v0} + 2)} + 2(\Gamma_{v0} + 1) \right].$$

Then relation (36) can be presented in the form of the Burgers' equation

$$\frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \varsigma} + \bar{\alpha} \frac{\partial^2 u_1}{\partial \varsigma^2} = 0,$$

$$\bar{\alpha} \equiv \frac{\varepsilon^{(\alpha-1)}}{2b\tau_{TP}} [b\tau_{TV} \tau_{TP} c^2 - (1 + \alpha_\infty T_0 \Gamma_{v0} \sigma(S)) v_0^2].$$

3. Evolution of nonlinear waves: the complete temporally nonlocal dynamical equation of state with the first order time derivatives and the nonlinear dynamical part

Suppose that the travelling wave propagates in media with the following equation of state

$$\begin{aligned} -\frac{1}{\tau_{TP}} \left[\frac{(\gamma_0 - 1)(\Gamma_{v0} + 1)}{\Gamma_{v0}} v_0^{(\Gamma_{v0}+2)} \sigma(S) - \frac{(\gamma_0 - 1)}{3\Gamma_{v0}} v_0^2 v^{\Gamma_{v0}} \right] \frac{\partial v}{\partial t} + \omega_0^2 v_0^{\Gamma_{v0}} \sigma(S) v - \\ - \frac{2}{3} \omega_0^2 v_0^{-1} v^{(\Gamma_{v0}+2)} - \frac{\omega_0^2}{3} v^{(\Gamma_{v0}+1)} = b v^{(\Gamma_{v0}+2)} (p - p_0) + b\tau_{TV} v^{(\Gamma_{v0}+2)} \frac{\partial p}{\partial t}. \end{aligned} \quad (37)$$

which forms the closed system together with equation (25).

Analogous to the previous section we are going to study the propagation of the small disturbance

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots, \quad v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots, \quad u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \quad (38)$$

Taking into account transformation $\xi = \varepsilon^\alpha (r - ct)$, $\theta = a\varepsilon^{(\alpha+1)}t$

$$\frac{\partial}{\partial t} = a\varepsilon^{(\alpha+1)} \frac{\partial}{\partial \theta} - c\varepsilon^\alpha \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial r} = \varepsilon^\alpha \frac{\partial}{\partial \xi},$$

expansions in series (38), and the formula

$$(v_0 + \varepsilon v_1 + \varepsilon^2 v_2)^n \approx v_0^n + n v_0^{(n-1)} \varepsilon v_1 + n v_0^{(n-1)} \varepsilon^2 v_2 + \frac{(n-1)n}{2} v_0^{[n-2]} \varepsilon^2 v$$

in the first order approximation the initial model is reduced to the following relations

$$v_1 = -\frac{v_0}{c} u_1, \quad p_1 = \frac{c}{v_0} u, \quad (39)$$

and

$$\begin{aligned} \omega_0^2 v_1 \sigma(S) - \frac{2}{3} \omega_0^2 (\Gamma_{v0} + 2) v_1 - \frac{\omega_0^2}{3} (\Gamma_{v0} + 1) v_1 &= b v_0^2 p_1, \Rightarrow \\ \sigma(S) \neq 1 : -\omega_0^2 v_1 \left(\Gamma_{v0} + \frac{5}{3} - \sigma(S) \right) &= b v_0^2 p_1; \\ \sigma(S) = 1 : -\omega_0^2 v_1 \left(\Gamma_{v0} + \frac{2}{3} \right) &= b v_0^2 p_1. \end{aligned}$$

Since

$$\frac{\omega_0^2}{b} = \frac{c_{s0}^2 \alpha_0 T_0}{\gamma_0},$$

then

$$\begin{cases} \left(\Gamma_{v0} + \frac{2}{3} \right) \frac{c_{s0}^2 \alpha_0 T_0}{\gamma_0} = -\frac{v_0^2 p_1}{v_1}, & \sigma(S) = 1; \\ \left(\Gamma_{v0} + \frac{5}{3} - \sigma(S) \right) \frac{c_{s0}^2 \alpha_0 T_0}{\gamma_0} = -\frac{v_0^2 p_1}{v_1}, & \sigma(S) \neq 1. \end{cases} \quad (40)$$

Inserting (39) into (40), we get the expression for velocity of small disturbance propagation in media

$$c^2 = \begin{cases} \left(\Gamma_{v0} + \frac{2}{3} \right) \frac{c_{s0}^2 \alpha_0 T_0}{\gamma_0} = \left(\Gamma_{v0} + \frac{2}{3} \right) \frac{\omega_0^2}{b}, & \sigma(S) = 1; \\ \left(\Gamma_{v0} + \frac{5}{3} - \sigma(S) \right) \frac{c_{s0}^2 \alpha_0 T_0}{\gamma_0} = - \left(\Gamma_{v0} + \frac{5}{3} - \sigma(S) \right) \frac{\omega_0^2}{b}, & \sigma(S) \neq 1, \end{cases} \quad (41)$$

with equation of state

$$p - p_0 = \frac{c_{s0}^2}{\gamma_0 v_0} \left\{ \frac{(1 - \gamma_0)}{3\Gamma_{v0}} \left[\left(\frac{v}{v_0} \right)^{-1} - 1 \right] + \frac{(\gamma_0 - 1)}{\Gamma_{v0}} \left[\sigma(S) \left(\frac{v}{v_0} \right)^{-(\Gamma_{v0}+1)} - 1 \right] \right\} -$$

$$-\tau_{TV} \frac{dp}{dt} - \frac{c_{s0}^2}{\gamma_0 v_0} \left[\frac{(1 - \gamma_0) v_0}{3\Gamma_{v0} v^2} + \frac{(\gamma_0 - 1)}{\Gamma_{v0}} \sigma(S) \frac{(\Gamma_{v0} + 1) v_0^{(\Gamma_{v0}+1)}}{v^{(\Gamma_{v0}+2)}} \right] \tau_{TP} \frac{dv}{dt}. \quad (42)$$

In the second order approximation

$$\frac{\partial v_2}{\partial \xi} = \frac{a}{c} \frac{\partial v_1}{\partial \theta} - \frac{v_0}{c} \frac{\partial u_2}{\partial \xi}, \quad \frac{\partial p_2}{\partial \xi} = -\frac{a}{v_0} \frac{\partial u_1}{\partial \theta} + \frac{c}{v_0} \frac{\partial u_2}{\partial \xi}. \quad (43)$$

and

$$-\frac{\varepsilon^{(\alpha-1)}}{\tau_{TP}} \left[\frac{(\gamma_0 - 1) (\Gamma_{v0} + 1)}{\Gamma_{v0}} v_0^{(\Gamma_{v0}+2)} \sigma(S) - \frac{(\gamma_0 - 1)}{3\Gamma_{v0}} v_0^{(\Gamma_{v0}+2)} \right] c \frac{\partial v_1}{\partial \xi} +$$

$$+\omega_0^2 v_0^{\Gamma_{v0}} \sigma(S) v_2 - \omega_0^2 v_0^{-1} (\Gamma_{v0} + 2) v_0^{(\Gamma_{v0}+1)} v_2 - \omega_0^2 v_0^{-1} \frac{(\Gamma_{v0} + 1) (\Gamma_{v0} + 2)}{2} v_0^{\Gamma_{v0}} v_1^2 -$$

$$-\frac{\omega_0^2}{3} (\Gamma_{v0} + 1) v_0^{\Gamma_{v0}} v_2 - \frac{\omega_0^2 \Gamma_{v0} (\Gamma_{v0} + 1)}{3} v_0^{(\Gamma_{v0}-1)} v_1^2 + \quad (44)$$

$$+\frac{\omega_0^2}{3} v_0^{-1} (\Gamma_{v0} + 2) v_0^{(\Gamma_{v0}+1)} v_2 + \frac{\omega_0^2}{3} v_0^{-1} \frac{(\Gamma_{v0} + 1) (\Gamma_{v0} + 2)}{2} v_0^{\Gamma_{v0}} v_1^2 =$$

$$= b (\Gamma_{v0} + 2) v_0^{(\Gamma_{v0}+1)} v_1 p_1 + b v_0^{(\Gamma_{v0}+2)} p_2 - b \tau_{TV} v_0^{(\Gamma_{v0}+2)} c \varepsilon^{(\alpha-1)} \frac{\partial p_1}{\partial \xi}.$$

Dividing equation (44) into $v_0^{(\Gamma_{v0}+2)}$, differentiating with respect to ξ , and inserting expressions (39), (43), we get

$$-\frac{(\gamma_0 - 1)}{\tau_{TP} \Gamma_{v0}} \left[(\Gamma_{v0} + 1) \sigma(S) - \frac{1}{3} \right] \varepsilon^{(\alpha-1)} v_0 \frac{\partial^2 u_1}{\partial \xi^2} -$$

$$-\frac{\omega_0^2}{v_0^2} \left[\Gamma_{v0} + \frac{5}{3} - \sigma(S) \right] \left[-\frac{a v_0}{c^2} \frac{\partial u_1}{\partial \theta} - \frac{v_0}{c} \frac{\partial u_2}{\partial \xi} \right] -$$

$$-\frac{\omega_0^2}{v_0 c^2} \frac{(\Gamma_{v0} + 1) (3\Gamma_{v0} + 4)}{3} u_1 \frac{\partial u_1}{\partial \xi} = -2 \frac{b (\Gamma_{v0} + 2)}{v_0} u_1 \frac{\partial u_1}{\partial \xi} + \quad (45)$$

$$+b \left[-\frac{a}{v_0} \frac{\partial u_1}{\partial \theta} + \frac{c}{v_0} \frac{\partial u_2}{\partial \xi} \right] - b\tau_{TV} \frac{c^2}{v_0} \varepsilon^{(\alpha-1)} \frac{\partial^2 u_1}{\partial \xi^2}.$$

Using

$$\frac{\omega_0^2}{v_0 c^2} \left[\Gamma_{V0} + \frac{5}{3} - \sigma(S) \right] a \frac{\partial u_1}{\partial \theta} = \frac{ab}{v_0} \frac{\partial u_1}{\partial \theta}; \frac{\omega_0^2}{v_0 c} \left[\Gamma_{V0} + \frac{5}{3} - \sigma(S) \right] \frac{\partial u_2}{\partial \xi} = \frac{bc}{v_0} \frac{\partial u_2}{\partial \xi};$$

$$\frac{\omega_0^2}{v_0 c^2} \frac{(\Gamma_{V0} + 1)(3\Gamma_{V0} + 4)}{3} = \frac{b}{v_0} \frac{(\Gamma_{V0} + 1)(3\Gamma_{V0} + 4)}{3 \left[\Gamma_{V0} + \frac{5}{3} - \sigma(S) \right]};$$

$$\left[-\frac{(\gamma_0 - 1)}{\tau_{TP} \Gamma_{V0}} \left[\Gamma_{V0} + \frac{5}{3} - \sigma(S) \right] v_0 + b\tau_{TV} \frac{c^2}{v_0} \right] = c^2 \left(-\frac{b\tau_{TP}}{v_0} + \frac{b\tau_{TV}}{v_0} \right),$$

equation (45) can be written in the form

$$2a \frac{\partial u_1}{\partial \theta} + \left[-\frac{(\Gamma_{V0} + 1)(3\Gamma_{V0} + 4)}{3 \left[\Gamma_{V0} + \frac{5}{3} - \sigma(S) \right]} + 2(\Gamma_{V0} + 2) \right] u_1 \frac{\partial u_1}{\partial \xi} + (\tau_{TV} - \tau_{TP}) c^2 \varepsilon^{(\alpha-1)} \frac{\partial^2 u_1}{\partial \xi^2} = 0. \quad (46)$$

Let

$$a \equiv \frac{1}{2} \left[-\frac{(\Gamma_{V0} + 1)(3\Gamma_{V0} + 4)}{3 \left[\Gamma_{V0} + \frac{5}{3} - \sigma(S) \right]} + 2(\Gamma_{V0} + 2) \right],$$

then equation (46) is reduced to the Burger's equation

$$\frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \xi} + \bar{\alpha} \frac{\partial^2 u_1}{\partial \xi^2} = 0, \quad \bar{\alpha} \equiv \frac{\varepsilon^{(\alpha-1)}}{2a} c^2 (\tau_{TV} - \tau_{TP}).$$

4. Evolution of nonlinear waves: the dynamical equation of state for media with temporal and spatial nonlocalities (the second order time derivatives)

In this section we are going to study the propagation of travelling waves in the nonlinear one dimensional media with temporal and spatial nonlocalities

and the second order time derivatives. In the Lagrange coordinates (r, t) the system of equations describing dynamics of nonlocal media takes the form

$$\begin{aligned}
 \frac{\partial v}{\partial t} - v_0 \frac{\partial u}{\partial r} &= 0, \quad \frac{1}{v_0} \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0, \\
 \frac{\Gamma \varepsilon_r}{\tau_{TP}} [v_0^2 v^{(\Gamma v_0+3)} v_{rr} - 2v_0^2 v^{(\Gamma v_0+2)} v_r^2] &+ \left[\frac{\partial^2 v}{\partial t^2} - \frac{1}{\tau_{TP}} \frac{\partial v}{\partial t} \right] v^{(\Gamma v_0+3)} + \\
 &+ \omega_0^2 v_0^{(\Gamma v_0-1)} v^5 - \omega_0^2 v_0^{-1} v^{(\Gamma v_0+5)} = \\
 &= \left[b(p - p_0) + b\tau_{TV} \frac{\partial p}{\partial t} - b\tau_{TP}\tau_{TV} \frac{\partial^2 p}{\partial t^2} \right] v^{(\Gamma v_0+5)} - \\
 &- \Gamma \varepsilon_r b\tau_{TV} [v_0^2 v^{(\Gamma v_0+3)} p_{rr} - 2v_0^2 v^{(\Gamma v_0+2)} v_r p_r], \\
 b &\equiv v_0 \chi_{T\infty} / \tau_{TV} \tau_{TP}, \omega_0^2 \equiv bc_{S0}^2 \alpha_0 T_0 / \gamma_0.
 \end{aligned} \tag{47}$$

Application of the perturbation theory

$$\begin{aligned}
 p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + O(\varepsilon^3), \quad v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + O(\varepsilon^3), \\
 u &= \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)
 \end{aligned}$$

and the scale transformations

$$\zeta = \varepsilon^\delta (r - ct), \quad \theta = a\varepsilon^{(\delta+1)} t, \quad a = \frac{(\Gamma v_0 + 1)}{2}$$

(here, c is the velocity of disturbance propagation, $\delta > 0$, a are parameters) are reduced the initial model in the first approximation to relations

$$v_1 = -\frac{v_0}{c} u_1, \quad p_1 = \frac{c}{v_0} u_1, \quad c^2 = \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0} \Gamma v_0 = \frac{c_{S0}^2 (\gamma_0 - 1)}{\gamma_0}.$$

In the second order approximation the initial system of partial differential equations is reduced to the single third order KdVB equation

$$\frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \zeta} + \alpha \frac{\partial^2 u_1}{\partial \zeta^2} + \beta \frac{\partial^3 u_1}{\partial \zeta^3} = 0, \tag{48}$$

$$\alpha \equiv \frac{\varepsilon^{(\delta-1)}}{b(\Gamma_{V0} + 1)} \frac{1}{\tau_{TP}} (b\tau_{TP}\tau_{TV}c^2 - v_0^2),$$

$$\beta \equiv \frac{\varepsilon^{(2\delta-1)}}{b(\Gamma_{V0} + 1)} \left(c + \frac{\Gamma\varepsilon_r}{\tau_{TP}c} \right) (b\tau_{TP}\tau_{TV}c^2 - v_0^2).$$

Amplitude equation (48) coincides with amplitude equation (12). Comparing the results of the investigations from sections 1. and 4., we make the following notation. Studying the model with the nonlinear nonlocal dynamical equation of state with the second order derivatives we obtain the third order nonlinear amplitude equation with the KdVB nonlinearity. If the second order time derivatives are omitted in the dynamical equation of state of system (47), then we pass to the problem studied in section 1. Thus, coefficient β in evolution equation (48) takes the form

$$\beta \equiv \frac{\varepsilon^{(2\delta-1)}}{b(\Gamma_{V0} + 1)} \frac{\Gamma\varepsilon_r}{\tau_{TP}c} (b\tau_{TP}\tau_{TV}c^2 - v_0^2).$$

5. Evolution of nonlinear waves: the nonlinear dynamical equation of state for media with spatial and temporal nonlocalities (the third order time derivatives)

In this section we study the nonlinear wave fields in media within the framework of mathematical model [4]

$$\begin{aligned} \frac{\partial v}{\partial t} - v_0 \frac{\partial u}{\partial r} &= 0, \quad \frac{1}{v_0} \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0, \\ \frac{\Gamma\varepsilon_r}{\tau_{TP}} [v_0^2 v^{(\Gamma_{V0}+3)} v_{rr} - 2v_0^2 v^{(\Gamma_{V0}+2)} v_r^2] &+ \left[-\tau_{TP} \frac{\partial^3 v}{\partial t^3} + \frac{\partial^2 v}{\partial t^2} - \right. \\ &\left. - \frac{1}{\tau_{TP}} \frac{\partial v}{\partial t} \right] v^{(\Gamma_{V0}+3)} + \omega_0^2 v_0^{(\Gamma_{V0}-1)} v^5 - \omega_0^2 v_0^{-1} v^{(\Gamma_{V0}+5)} = \\ &= \left[b(p - p_0) + b\tau_{TV} \frac{\partial p}{\partial t} - b\tau_{TP}\tau_{TV} \frac{\partial^2 p}{\partial t^2} + b\tau_{TP}^2 \tau_{TV} \frac{\partial^3 p}{\partial t^3} \right] v^{(\Gamma_{V0}+5)} - \\ &- \Gamma\varepsilon_r b\tau_{TV} [v_0^2 v^{(\Gamma_{V0}+3)} p_{rr} - 2v_0^2 v^{(\Gamma_{V0}+2)} v_r p_r], \end{aligned} \quad (49)$$

where (r, t) are the Lagrange spatial coordinates,

$$b \equiv v_0 \chi_{T\infty} / \tau_{TV} \tau_{TP}, \omega_0^2 \equiv b c_{S0}^2 \alpha_0 T_0 / \gamma_0.$$

The application of perturbation theory (4) and scales transformations (5) to initial system (49) yeild the following relations in the first order approximation

$$v_1 = -\frac{v_0}{c} u_1, \quad p_1 = \frac{c}{v_0} u_1, \quad c^2 = \frac{c_{S0}^2 (\gamma_0 - 1)}{\gamma_0}$$

These relations coincide with (8)-(9) obtained for media with the dynamical equation of state (2) with the first order time derivatives and the second order spatial derivatives.

In the second order approximation of the perturbation theory, the reduction from the initial system of partial differential equations (49) to the single nonlinear fourth order equation were carried out

$$\frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \zeta} + \alpha \frac{\partial^2 u_1}{\partial \zeta^2} + \beta \frac{\partial^3 u_1}{\partial \zeta^3} + \gamma \frac{\partial^4 u_1}{\partial \zeta^4} = 0, \quad (50)$$

$$\begin{aligned} \alpha &\equiv \frac{\varepsilon^{(\delta-1)}}{b(\Gamma_{V0} + 1) \tau_{TP}} (b \tau_{TP} \tau_{TV} c^2 - v_0^2), \\ \beta &\equiv \frac{\varepsilon^{(2\delta-1)}}{b(\Gamma_{V0} + 1)} \left(c + \frac{\Gamma \varepsilon_r}{\tau_{TP} c} \right) (b \tau_{TP} \tau_{TV} c^2 - v_0^2), \\ \gamma &\equiv \frac{\varepsilon^{(3\delta-1)}}{b(\Gamma_{V0} + 1)} \tau_{TP} c^2 (b \tau_{TP} \tau_{TV} c^2 - v_0^2). \end{aligned} \quad (51)$$

Amplitude equation (50) is the nonlinear fourth order equation with KdVB nonlinearity. It is well known as the Kuramoto-Sivashinski equation belonging to the nonintegrable equations. Using the method of singular manifolds offered in [3] some solutions of this equation can be derived if coefficients (51) satisfy the relations

$$\beta^2 = n \alpha \gamma, \quad n = 16, \quad n = \frac{144}{47}, \quad n = \frac{256}{73}. \quad (52)$$

Condition (52) holds, if the parameter of spatial nonlocality ε_r is the root of the quadratic equation

$$\frac{\Gamma^2}{\tau_{TP}^2} (\varepsilon_r)^2 + \frac{2c^2 \Gamma}{\tau_{TP}} \varepsilon_r + c^4 (1 - n) = 0.$$

Since $\varepsilon_r > 0$, then

$$\varepsilon_r = \frac{c^2 \tau_{\text{TP}}}{\Gamma} (\sqrt{n} - 1).$$

Spatial scale l_ε is connected with the parameter of spatial nonlocality ε_r by means of relation $l_\varepsilon^2 = \Gamma \varepsilon_r \tau_{\text{TV}}$. As shown in [3], if relations (52) hold, then equation (50) has the exact analytical solution

$$u_1(\zeta, \theta) = C_1 + \left[\frac{15}{76} \left(16\alpha - \frac{\beta^2}{\gamma} \right) k + 15\beta k^2 + 60\gamma k^3 \right] \frac{E}{C_2 + E} - (53)$$

$$- (15\beta k^2 + 180\gamma k^3) \frac{E^2}{(C_2 + E)^2} + 60\gamma k^3 \frac{E^3}{(C_2 + E)^3},$$

where

$$E = \exp(k\zeta + \omega\theta), \quad C_{1,2} = \text{const};$$

$$n = 16, \quad k = \pm \sqrt{\alpha/\gamma}, \quad \omega = C_1 k + \frac{3\beta}{2} k^3;$$

$$n = \frac{144}{47}, \quad k = \pm \sqrt{\alpha/47\gamma}, \quad \omega = -C_1 k - \frac{60\alpha}{47} k^2;$$

$$n = \frac{256}{73}, \quad k = \pm \sqrt{\alpha/73\gamma}, \quad \omega = -C_1 k - \frac{90\alpha}{73} k^2.$$

Thus, for nonlinear relaxing media with spatial and temporal nonlocalities at the following values of parameter ε_r

$$(\varepsilon_r)_1 = \frac{3c^2 \tau_{\text{TP}}}{\Gamma} \quad \text{for } n = 16;$$

$$(\varepsilon_r)_2 = \left(\sqrt{\frac{144}{47}} - 1 \right) \frac{c^2 \tau_{\text{TP}}}{\Gamma} \quad \text{for } n = \frac{144}{47};$$

$$(\varepsilon_r)_3 = \left(\sqrt{\frac{256}{73}} - 1 \right) \frac{c^2 \tau_{\text{TP}}}{\Gamma} \quad \text{for } n = \frac{256}{73}$$

spatial scales exist

$$(l_\varepsilon)_1 = c\sqrt{3\tau_{\text{TP}}\tau_{\text{TV}}} \quad \text{for } n = 16;$$

$$(l_\varepsilon)_2 = c \sqrt{\left(\sqrt{\frac{144}{47}} - 1\right) \tau_{\text{TP}} \tau_{\text{TV}}} \quad \text{for } n = \frac{144}{47};$$

$$(l_\varepsilon)_3 = c \sqrt{\left(\sqrt{\frac{256}{73}} - 1\right) \tau_{\text{TP}} \tau_{\text{TV}}} \quad \text{for } n = \frac{256}{73},$$

on which the wave amplitude is described by function (53). If $n = 16$, $k = \sqrt{\alpha/\gamma}$, $\omega = 6\alpha^2/\gamma$, $C_1 = 0$, $C_2 = 1$, then from (53) it follows the solution of equation (50) [3]:

$$u_1(\zeta, \theta) = 15\alpha\sqrt{\alpha/\gamma} \frac{\left[1 - th \left\{ \frac{\sqrt{\alpha/\gamma}}{2} (\zeta - 6\alpha\sqrt{\alpha/\gamma}\theta) \right\}\right]}{ch^2 \left\{ \frac{\sqrt{\alpha/\gamma}}{2} (\zeta - 6\alpha\sqrt{\alpha/\gamma}\theta) \right\}}. \quad (54)$$

This solution is the solitary wave with single maximum $(160\alpha/9)\sqrt{\alpha/\gamma}$. Taking the limit $Z \rightarrow \pm\infty$ ($Z \equiv \zeta - 6\alpha\sqrt{\alpha/\gamma}\theta$), we see that $u_1(Z) \rightarrow 0$. As shown in [3], solitary wave (54) is the classical soliton.

6. Evolution of nonlinear waves: the nonlinear nonlocal dynamical equation of state with the third order time derivatives taking into account the temperature terms and nonstationarity of thermodynamical forces

In this section the exact soliton solutions to the nonlinear equations for relaxing media taking into account the temperature and nonstationary generalized thermodynamical forces are studied. We use the following mathematical model

$$\frac{\partial v}{\partial t} - v_0 \frac{\partial u}{\partial r} = 0, \quad \frac{1}{v_0} \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0,$$

$$\begin{aligned}
& -\Gamma\epsilon_r \left\{ \left\{ 6v^{(\Gamma_{V0}+1)} + [2 + 3(\Gamma_{V0} + 2) + (\Gamma_{V0} + 2)^2] \theta \right\} v_0^2 v_t (v_r)^2 - \right. \\
& - [4v^{(\Gamma_{V0}+2)} + 2(\Gamma_{V0} + 3) \theta v] v_0^2 v_r v_{rt} - [2v^{(\Gamma_{V0}+2)} + (\Gamma_{V0} + 3) \theta v] v_0^2 v_t v_{rr} + \\
& + [v^{(\Gamma_{V0}+3)} + \theta v^2] v_0^2 v_{rrt} \left. \right\} + \frac{\Gamma\epsilon_r}{\tau_{TP}} \left\{ [v^{(\Gamma_{V0}+3)} + \theta v^2] v_0^2 v_{rr} - \right. \\
& - [2v^{(\Gamma_{V0}+2)} + (\Gamma_{V0} + 3) \theta v] v_0^2 (v_r)^2 \left. \right\} - \quad (55) \\
& -\tau_{TP} \left\{ [v^{(\Gamma_{V0}+5)} + \theta v^4] \frac{\partial^3 v}{\partial t^3} - 3(\Gamma_{V0} + 1) \theta v^3 \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial t^2} + \right. \\
& + (\Gamma_{V0} + 1)(\Gamma_{V0} + 2) \theta v^2 \left(\frac{\partial v}{\partial t} \right)^3 \left. \right\} + [v^{(\Gamma_{V0}+5)} + \theta v^4] \frac{\partial^2 v}{\partial t^2} - \\
& - (\Gamma_{V0} + 1) \theta v^3 \left(\frac{\partial v}{\partial t} \right)^2 - \frac{1}{\tau_{TP}} \left[\left(1 - \frac{\alpha_0}{\alpha_\infty} \right) \theta v^4 \right] \frac{\partial v}{\partial t} + \\
& + \omega_0^2 v_0^{(\Gamma_{V0}-1)} v^5 - \omega_0^2 v_0^{-1} v^{(\Gamma_{V0}+5)} = \\
& = b v^{(\Gamma_{V0}+5)} (p - p_0) + b(\tau_{TV} - \tau_{TP}) v^{(\Gamma_{V0}+5)} \frac{\partial p}{\partial t} - b \tau_{TP} \tau_{TV} v^{(\Gamma_{V0}+5)} \frac{\partial^2 p}{\partial t^2} + \\
& + b \tau_{TP}^2 \tau_{TV} v^{(\Gamma_{V0}+5)} \frac{\partial^3 p}{\partial t^3} - \Gamma\epsilon_r b \tau_{TV} [v_0^2 v^{(\Gamma_{V0}+3)} p_{rr} - 2v_0^2 v^{(\Gamma_{V0}+2)} v_r p_r] + \\
& + \Gamma\epsilon_r b \tau_{TP} \tau_{TV} [6v_0^2 v^{(\Gamma_{V0}+1)} v_t v_r p_r - 2v_0^2 v^{(\Gamma_{V0}+2)} v_{rt} p_r - \\
& - 2v_0^2 v^{(\Gamma_{V0}+2)} v_r p_{rt} - 2v_0^2 v^{(\Gamma_{V0}+2)} v_t p_{rr} + v_0^2 v^{(\Gamma_{V0}+3)} p_{rrt}] ,
\end{aligned}$$

where (r, t) are the Lagrange coordinates, $\theta = \alpha_\infty T_0 \Gamma_{V0} v_0^{(\Gamma_{V0}+1)}$, $b = v_0 \chi_{T\infty} / \tau_{TV} \tau_{TP}$, $\omega_0^2 = b c_{S0}^2 \alpha_0 T_0 / \gamma_0$.

The third equation of system (55) is equation of state [5]

$$\begin{aligned}
& \frac{1}{\rho^2} \frac{\Gamma\epsilon_r}{\tau_{TP}} \left\{ \left[-\rho_{xx} (1 - a) + \frac{1}{\rho} (\rho_x)^2 (1 - a \Gamma_{V0}) \right] + \left[-\frac{d^2 \rho}{dt^2} (1 + a) + \right. \right. \\
& + \frac{2}{\rho} \left(\frac{d\rho}{dt} \right)^2 \left(1 - \frac{a(\Gamma_{V0} - 1)}{2} \right) + \frac{1}{\tau_{TP}} \frac{d\rho}{dt} (1 + a) \left. \right] + \frac{\tau_{TP}}{\rho^2} \frac{d^3 \rho}{dt^3} \left(1 + \left(\frac{\alpha_0}{\alpha_\infty} \right)^3 a \right) - \\
& - \frac{3}{\rho} \left(2 - \left(\frac{\alpha_0}{\alpha_\infty} \right)^3 (\Gamma_{V0} + 1) a \right) \frac{d\rho}{dt} \frac{d^2 \rho}{dt^2} + \quad (56)
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\rho^2} \left(6 - \left(\frac{\alpha_0}{\alpha_\infty} \right)^3 (\Gamma_{V0} + 1) (\Gamma_{V0} + 2) a \right) \left(\frac{d\rho}{dt} \right)^3 + \omega_0^2 \rho_0^{1-\Gamma_{V0}} \rho_0^{\Gamma_{V0}} - \omega_0^2 \rho_0 = \\
 & = b(p - p_0) + b_{TV} \frac{dp}{dt} - \frac{\chi_{T0}}{\chi_{T\infty}} b \tau_{TV}^2 \frac{d^2 p}{dt^2} + \left(\frac{\chi_{T0}}{\chi_{T\infty}} \right)^2 b \tau_{TV}^3 \frac{d^3 p}{dt^3} - b \Gamma \varepsilon_r \tau_{TV} \left(p_{xx} + \frac{\rho_x}{\rho} p_x \right),
 \end{aligned}$$

given in coordinates (r, t) . This equation contains the temperature terms due to application of relation

$$T - T_0 = T_0 \left[e^{\frac{S-S_0}{C_{V\infty}}} \left(\frac{v}{v_0} \right)^{-\Gamma_{V\infty}} - 1 \right], \quad (57)$$

written for T and v in the adiabatic approximation ($S = \text{const}$, $\sigma(S) = 1$, $\Gamma_{V\infty} = \Gamma_v$).

The procedure of reduction from system (55) to a single high order nonlinear equation is similar to the one presented in the previous sections. In the first order approximation of the perturbation theory from system (55), we obtain the relations between the perturbed values and the expression for the velocity of disturbance perturbation. In the second order approximation, the amplitude evolution equation is derived

$$\frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \zeta} + \alpha \frac{\partial^2 u_1}{\partial \zeta^2} + \beta \frac{\partial^3 u_1}{\partial \zeta^3} + \gamma \frac{\partial^4 u_1}{\partial \zeta^4} = 0, \quad (58)$$

$$\begin{aligned}
 \alpha & \equiv \frac{\varepsilon^{(\delta-1)}}{b(\Gamma_{V0} + 1) \tau_{TP}} \left[b \tau_{TP} \tau_{TV} c^2 - b \tau_{TP}^2 c^2 - (\alpha_\infty - \alpha_0) T_0 \Gamma_{V0} v_0^2 \right], \\
 \beta & \equiv \frac{\varepsilon^{(2\delta-1)}}{b(\Gamma_{V0} + 1)} \left(c + \frac{\Gamma \varepsilon_r}{\tau_{TP} c} \right) \left[b \tau_{TP} \tau_{TV} c^2 - (1 + \alpha_\infty T_0 \Gamma_{V0}) v_0^2 \right], \\
 \gamma & \equiv \frac{\varepsilon^{(3\delta-1)}}{b(\Gamma_{V0} + 1) \tau_{TP}} \left(c^2 + \frac{\Gamma \varepsilon_r}{\tau_{TP}} \right) \left[b \tau_{TP} \tau_{TV} c^2 - (1 + \alpha_\infty T_0 \Gamma_{V0}) v_0^2 \right], \\
 c^2 & = \frac{c_{S0}^2 (\gamma_0 - 1)}{\gamma_0}.
 \end{aligned} \quad (59)$$

The including of the temperature and nonstationary thermodynamical forces does not change amplitude equation (58) in comparison with the previous investigations, but coefficients (59) have changed. The exact solution of equation (58) has the form (53).

Taking into account relations (52), (59) we obtain the following condition for the parameter of spatial nonlocality

$$\varepsilon_r = \frac{\tau_{\text{TP}} c^2}{\Gamma} \left[(n-1) - n \frac{f}{D} \right],$$

$$f \equiv b \tau_{\text{TP}}^2 c^2 - (1 + \alpha_0 T_0 \Gamma v_0) v_0^2; \quad D \equiv b \tau_{\text{TP}} \tau_{\text{TV}} c^2 - (1 + \alpha_\infty T_0 \Gamma v_0) v_0^2.$$

Thus, at the values of the parameter of spatial nonlocality

$$(\varepsilon_r)_1 = \frac{\tau_{\text{TP}} c^2}{\Gamma} \left(15 - 16 \frac{f}{D} \right) \quad \text{for } n = 16;$$

$$(\varepsilon_r)_2 = \frac{\tau_{\text{TP}} c^2}{47\Gamma} \left(97 - 144 \frac{f}{D} \right) \quad \text{for } n = \frac{144}{47};$$

$$(\varepsilon_r)_3 = \frac{\tau_{\text{TP}} c^2}{73\Gamma} \left(183 - 256 \frac{f}{D} \right) \quad \text{for } n = \frac{256}{73}$$

the small disturbance u_1 propagated as a travelling wave slowly varying develops in accordance with equation (58) and its exact solution (53).

7. Evolution of nonlinear waves: the third order dynamical equation of state taking into account elastic and thermal pressure components

Consider the evolution of the small disturbance in media with dynamical equation of state

$$-\frac{dp}{dt} - \frac{1}{v_0 \chi_{T0}} \frac{dv}{dt} + \frac{\alpha_0}{\chi_{T0}} \frac{dT}{dt}, \quad (60)$$

and find the influence of elastic pressure components on the results of evolutionary investigations. The mathematical model of nonlocal media has the form

$$\begin{aligned} & \omega_0^2 v_0^{(\Gamma v_0 - 1)} v^5 - \omega_0^2 v_0^{-1} v^{(\Gamma v_0 + 5)} - \tilde{\omega}_0^2 v_0^{-2} v^{(\Gamma v_0 + 6)} + \tilde{\omega}_0^2 v_0^{-1} v^{(\Gamma v_0 + 5)} = \\ & = b v^{(\Gamma v_0 + 5)} (p - p_0), \quad \omega_0^2 \equiv \frac{b c_{s0}^2 (\gamma_0 - 1)}{\gamma_0 \Gamma v_0}, \quad \tilde{\omega}_0^2 \equiv \frac{b c_{s0}^2}{\gamma_0}. \end{aligned}$$

Similar to the previous sections, we introduce the scales transformations $\zeta = \varepsilon^\delta(r - ct)$, $\theta = a\varepsilon^{(\delta+1)}t$ and use the methods of the perturbation theory. In the first order approximation, the perturbed values are described by the relations

$$v_1 = -\frac{v_0}{c}u_1, \quad p_1 = \frac{c}{v_0}u_1, \quad c^2 = c_{s0}^2.$$

It can be checked that the spatial nonlocality, temperature terms, and non-stationarity of thermodynamical forces do not influence velocity c of travelling wave propagation. Velocity c is defined by the quasistationary part of the dynamical equation of state.

In the quadratic order approximation of the perturbation theory, the initial model is reduced to system

$$\begin{aligned} \frac{\partial v_2}{\partial \zeta} &= \frac{a}{c} \frac{\partial v_1}{\partial \theta} - \frac{v_0}{c} \frac{\partial u_2}{\partial \zeta}, \quad \frac{\partial p_2}{\partial \zeta} = -\frac{a}{v_0} \frac{\partial u_1}{\partial \theta} + \frac{c}{v_0} \frac{\partial u_2}{\partial \zeta}, \\ \frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \zeta} + \alpha \frac{\partial^2 u_1}{\partial \zeta^2} + \beta \frac{\partial^3 u_1}{\partial \zeta^3} + \gamma \frac{\partial^4 u_1}{\partial \zeta^4} &= 0, \\ \alpha &\equiv \frac{\varepsilon^{(\delta-1)}}{2ab} \frac{1}{\tau_{\text{TP}}} [b\tau_{\text{TP}}\tau_{\text{TV}}c^2 - b\tau_{\text{TP}}^2 c^2 - (\alpha_\infty - \alpha_0) T_0 \Gamma_{v0} v_0^2], \\ \beta &\equiv \frac{\varepsilon^{(2\delta-1)}}{2abc} \left(c^2 + \frac{\Gamma \varepsilon_r}{\tau_{\text{TP}}} \right) [b\tau_{\text{TP}}\tau_{\text{TV}}c^2 - (1 + \alpha_\infty T_0 \Gamma_{v0}) v_0^2], \\ \gamma &\equiv \frac{\varepsilon^{(3\delta-1)}}{2ab} \tau_{\text{TP}} \left(c^2 + \frac{\Gamma \varepsilon_r}{\tau_{\text{TP}}} \right) [b\tau_{\text{TP}}\tau_{\text{TV}}c^2 - (1 + \alpha_\infty T_0 \Gamma_{v0}) v_0^2], \\ a &= \frac{(\gamma_0 - 1)(\Gamma_{v0} + 1)}{2\gamma_0}. \end{aligned}$$

8. Generalized evolutionary equations for media with temporal nonlocality

In this section we study the propagations of travelling wave in media which are characterized by phenomenological constitutive relation

$$\frac{d\xi}{dt} = \Gamma A + \tau_{\text{TP}} \frac{d^2 \xi}{dt^2} - \tau_{\text{TP}}^2 \frac{d^3 \xi}{dt^3} + \tau_{\text{TP}}^3 \frac{d^4 \xi}{dt^4}. \quad (61)$$

Then, the dynamical equation of state has the form

$$\begin{aligned}
 p - p_0 = & \frac{c_{S0}^2}{\gamma_0 \nu_0} \left(1 - \frac{\nu}{\nu_0} \right) + \frac{c_{S0}^2 \alpha_0 T_0}{\nu_0 \gamma_0} \left[\sigma(S) \left(\frac{\nu}{\nu_0} \right)^{-\Gamma_{\nu_0}} - 1 \right] - \\
 & - \tau_{TV} \frac{dp}{dt} - \frac{\tau_{TP}}{\nu_0 \chi_{T0}} \frac{d\nu}{dt} + \frac{\alpha_0}{\chi_{T0}} \tau_{PV} \frac{dT}{dt} + \left(\frac{\chi_{T0}}{\chi_{T\infty}} \right) \tau_{TV}^2 \frac{d^2 p}{dt^2} + \frac{\tau_{TP}^2}{\nu_0 \chi_{T0}} \frac{d^2 \nu}{dt^2} - \\
 & - \frac{\alpha_0^2}{\chi_{T0} \alpha_\infty} \tau_{PV}^2 \frac{d^2 T}{dt^2} - \left(\frac{\chi_{T0}}{\chi_{T\infty}} \right)^2 \tau_{TV}^3 \frac{d^3 p}{dt^3} - \frac{\tau_{TP}^3}{\nu_0 \chi_{T0}} \frac{d^3 \nu}{dt^3} + \frac{\alpha_0^3}{\chi_{T0} \alpha_\infty^2} \tau_{PV}^3 \frac{d^3 T}{dt^3} + \\
 & + \left(\frac{\chi_{T0}}{\chi_{T\infty}} \right)^3 \tau_{TV}^4 \frac{d^4 p}{dt^4} + \frac{\tau_{TP}^4}{\nu_0 \chi_{T0}} \frac{d^4 \nu}{dt^4} - \frac{\alpha_0^4}{\chi_{T0} \alpha_\infty^3} \tau_{PV}^4 \frac{d^4 T}{dt^4}.
 \end{aligned} \tag{62}$$

Using the same manipulations as in the previous sections we state that velocity c of the travelling wave in the first order approximation is equal to c_{S0}^2 .

If parameter $a = \frac{(\Gamma_{\nu_0} + 1)(\gamma_0 - 1)}{2\gamma_0}$ in the second order approximation, the evolutionary equation with respect to perturbed mass velocity u_1 has the form

$$\frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \zeta} + \alpha \frac{\partial^2 u_1}{\partial \zeta^2} + \beta \frac{\partial^3 u_1}{\partial \zeta^3} + \gamma \frac{\partial^4 u_1}{\partial \zeta^4} + \nu \frac{\partial^5 u_1}{\partial \zeta^5} = 0, \tag{63}$$

with coefficients

$$\begin{aligned}
 \alpha \equiv & \frac{\varepsilon^{(\delta-1)}}{2ab} \frac{1}{\tau_{TP}} (b\tau_{TP}\tau_{TV}c^2 - \nu_0^2), \quad \beta \equiv \frac{\varepsilon^{(2\delta-1)}}{2ab} c (b\tau_{TP}\tau_{TV}c^2 - \nu_0^2), \\
 \gamma \equiv & \frac{\varepsilon^{(3\delta-1)}}{2ab} \tau_{TP}c^2 (b\tau_{TP}\tau_{TV}c^2 - \nu_0^2), \quad \nu \equiv \frac{\varepsilon^{(4\delta-1)}}{2ab} \tau_{TP}^2c^3 (b\tau_{TP}\tau_{TV}c^2 - \nu_0^2),
 \end{aligned} \tag{64}$$

where $b \equiv \nu_0 \chi_{T\infty} / \tau_{TV} \tau_{TP}$. Equation (63) is the fifth order equation with partial derivatives and KdVB nonlinearity.

Now, consider the evolution of the small disturbance within the framework of generalized the N-th order dynamical equation of state

$$\begin{aligned}
 p - p_0 = & \frac{c_{S0}^2}{\gamma_0 \nu_0} \left(1 - \frac{\nu}{\nu_0} \right) + \frac{c_{S0}^2 \alpha_0 T_0}{\nu_0 \gamma_0} \left[\sigma(S) \left(\frac{\nu}{\nu_0} \right)^{-\Gamma_{\nu_0}} - 1 \right] - \\
 & - \sum_{n=0}^{N-1} (-1)^n \frac{d^n}{dt^n} \left[\left(\frac{\chi_{T0}}{\chi_{T\infty}} \right)^n \tau_{TV}^{(n+1)} \frac{dp}{dt} + \frac{\tau_{TP}^{(n+1)}}{\nu_0 \chi_{T0}} \frac{d\nu}{dt} - \frac{\alpha_0^{(n+1)}}{\chi_{T0} \alpha_\infty^n} \tau_{PV}^{(n+1)} \frac{dT}{dt} \right].
 \end{aligned} \tag{65}$$

for media with temporal nonlocality taking into account elastic and thermal pressure components

$$p - p_0 = \frac{c_{s0}^2}{\gamma_0 v_0} \left(1 - \frac{v}{v_0} \right) + \frac{c_{s0}^2 \alpha_0 T_0}{v_0 \gamma_0} \left[\sigma(S) \left(\frac{v}{v_0} \right)^{-\Gamma v_0} - 1 \right] - \sum_{n=0}^{N-1} (-1)^n \frac{d^n}{dt^n} \left[\left(\frac{\chi_{T0}}{\chi_{T\infty}} \right)^n \tau_{TV}^{(n+1)} \frac{dp}{dt} + \frac{\tau_{TP}^{(n+1)}}{v_0 \chi_{T0}} \frac{dv}{dt} - \frac{\alpha_0^{(n+1)}}{\chi_{T0} \alpha_\infty^n} \tau_{PV}^{(n+1)} \frac{dT}{dt} \right].$$

The corresponding evolutionary equation with respect to the perturbed mass velocity u_1 has the form

$$\frac{\partial u_1}{\partial \theta} + u_1 \frac{\partial u_1}{\partial \zeta} + \sum_{n=1}^N \alpha_n \frac{\partial^{(n+1)} u_1}{\partial \zeta^{(n+1)}} = 0,$$

with coefficients

$$\alpha_n = \frac{\varepsilon^{(n\delta-1)}}{2ab} \tau_{TP}^{(n-2)} c^{(n-1)} (b \tau_{TP} \tau_{TV} c^2 - v_0^2).$$

9. Investigations of localized wave solutions with the qualitative and numerical methods

Consider the wave solutions to equation (50) in the form $u_1 = U(y)$, $y = \kappa \zeta - \omega \theta$. Inserting the solution into (50) we obtain

$$-\omega U' + U \kappa U' + \alpha \kappa^2 U'' + \beta \kappa^3 U''' + \gamma \kappa^4 U'''' = 0.$$

where $(\cdot)' = \frac{d}{dy}(\cdot)$. Integrating the last equation we get

$$\frac{(U \kappa - \omega)^2}{2\kappa} + \alpha \kappa^2 U' + \beta \kappa^3 U'' + \gamma \kappa^4 U''' = L.$$

Point $U = 0$, $U' = U'' = U''' = 0$ is a fixed point so that $\frac{\omega^2}{2\kappa} = L \Rightarrow$

$$\frac{(U \kappa - \omega)^2}{2\kappa} + \alpha \kappa^2 U' + \beta \kappa^3 U'' + \gamma \kappa^4 U''' = \frac{\omega^2}{2\kappa}$$

or

$$\frac{U^2 \kappa}{2} - U\omega + \alpha \kappa^2 U' + \beta \kappa^3 U'' + \gamma \kappa^4 U''' = 0.$$

Then

$$U''' = -\frac{\alpha \kappa^2}{\gamma \kappa^4} U' - \frac{\beta \kappa^3}{\gamma \kappa^4} U'' + \frac{\omega}{\gamma \kappa^4} U - \frac{\kappa}{2\gamma \kappa^4} U^2$$

or

$$U''' = -\frac{\alpha}{\gamma \kappa^2} U' - \frac{\beta}{\gamma \kappa} U'' + \frac{\omega}{\gamma \kappa^4} U - \frac{1}{2\gamma \kappa^3} U^2. \quad (66)$$

To construct an exact solution of equation (66) or (50) we use the following relations

$$\beta^2 = \alpha \gamma n = 16\alpha \gamma, \kappa = \sqrt{\frac{\alpha}{\gamma}}, \omega = \frac{6\alpha^2}{\gamma}. \quad (67)$$

Then, under condition (67), the coefficients of differential equation (66) take the following values

$$\frac{\alpha}{\gamma \kappa^2} = 1, \quad \frac{\beta}{\gamma \kappa} = \frac{4\sqrt{\alpha \gamma}}{\gamma \sqrt{\alpha/\gamma}} = \frac{4\sqrt{\alpha \gamma}}{\sqrt{\alpha \gamma}} = 4, \quad \frac{\omega}{\gamma \kappa^4} = \frac{6\alpha^2}{\gamma^2 \alpha^2 \gamma^{-2}} = 6.$$

Thus, equation (66) under conditions (67) has the form

$$U''' = -U' - 4U'' + 6U - A_4 U^2, \quad A_4 = \frac{1}{2\gamma \kappa^3}. \quad (68)$$

The exact soliton solution of equation (68) can be written in the form

$$U = \frac{60}{A_4} \frac{e^y}{(1 + e^y)^3}.$$

Consider the mathematical model for media with temporal and spatial nonlocalities in the form

$$\begin{aligned} v_t - v_0 u_r &= 0, \quad u_t + v_0 p_r = 0, \\ \sigma v_0^2 [v v_{rr} - 2v_r^2] v^{n+2} + [-Q^2 \tau^2 v_{ttt} + Q \tau v_{tt} - v_t] v^{n+3} + \frac{\delta c^2}{n v_0} v^5 (v_0^n - v^n) &= \\ = \delta v^{n+5} [p - p_0 + \tau p_t - Q \tau^2 p_{tt} + Q^2 \tau^3 p_{ttt}] - \sigma \delta \tau v_0^2 [v p_{rr} - 2v_r p_r] v^{n+2}, \end{aligned} \quad (69)$$

where $\sigma = \Gamma \varepsilon_r$, $\tau_{TP} = Q \tau_{TV} = Q \tau$, $\Gamma_{V0} = n$, $bQ\tau = \delta$, $\omega_0^2 = b \frac{c^2}{n}$.

Let us analyze the structure of solutions

$$v = R(\xi), \quad u = U(\xi), \quad p = P(\xi), \quad \xi = Kr - Wt. \quad (70)$$

From the first equation of system (69) we get

$$WR + v_0 KU = L_1,$$

from the second equation it follows

$$-\frac{W}{v_0}U + PK = L_2.$$

For the stationary regime $u = 0$, $v = v_0$, $p = p_0$, constants $L_1 = Wv_0$, $L_2 = p_0K$, then

$$R = v_0 - v_0 \frac{K}{W}U, \quad P = p_0 + \frac{W}{v_0 K}U.$$

Inserting given expressions into the third equation of system (69), we obtain

$$U''' = A_1 U' + A_2 (U) U'^2 + A_3 (U) U'' + A_4 (U)$$

or

$$U' = Y, \quad Y' = Z, \quad Z' = A_1 Y + A_2 Y^2 + A_3 Z + A_4, \quad (71)$$

where

$$\begin{aligned} A_1 &= -\frac{1}{W^2 Q^2 \tau^2}, \quad A_2 = \frac{2K^3 \sigma (W^2 \delta \tau + K^2 v_0^2)}{(W - KU) W^3 Q^2 \tau^2 \{ \delta \tau (W - KU)^2 - K^2 \}}, \\ A_3 &= \frac{K^2 \tau (Q + \delta \sigma - \delta Q \tau U^2) W^2 + 2\delta K Q \tau^2 U W^3 - \delta Q \tau^2 W^4 + K^4 \sigma v_0^2}{W^3 Q^2 \tau^2 \{ \delta \tau (W - KU)^2 - K^2 \}}, \\ A_4 &= \frac{\delta (W - KU)^2 (v_0 - U \frac{K v_0}{W})^{-n} [-c^2 K v_0^n + (c^2 K + W n U) (v_0 - U \frac{K v_0}{W})^n]}{W^4 n Q^2 \tau^2 \{ \delta \tau (W - KU)^2 - K^2 \}}. \end{aligned}$$

Let us make the linearization of the coefficients of equation (71) in the vicinity of point $U = 0$ in the following way

$$\begin{aligned} A_2(U) &= A_2 + O(U) = \frac{2K^3\sigma(W^2\delta\tau + K^2v_0^2)}{W^4Q^2\tau^2(\delta\tau W^2 - K^2)} + O(U), \\ A_3(U) &= A_3 + O(U) = \frac{K^4\sigma v_0^2 + K^2\tau(Q + \delta\sigma)W^2 - \delta Q\tau^2W^4}{W^3Q^2\tau^2(\delta\tau W^2 - K^2)} + O(U), \\ A_4(U) &= A_{41}U + A_{42}U^2 + O(U^3) = \\ &= \frac{\delta(W^2 - K^2c^2)}{W^3Q^2\tau^2(\delta\tau W^2 - K^2)}U - \frac{\delta K^3(c^2(K^2 + \delta\tau W^2) - 2W^2)}{W^4Q^2\tau^2(\delta\tau W^2 - K^2)^2}U^2 + O(U^3), \end{aligned}$$

where $A_i = \text{const}$, and omit the term $A_2U'^2$ as a nonlinearity of a higher order. Then the linearized system corresponding to (71), has the form

$$U''' = A_1U' + A_3U'' + A_{41}U + A_{42}U^2. \quad (72)$$

To reduce equation (72) to (68) we must impose the conditions

$$A_1 = -1, A_3 = -4, A_{41} = 6.$$

These conditions are satisfied under the following values of parameters

$$\begin{aligned} b &= 0.5, \quad K = 0.8, \quad c = 1, \quad n = 1, \quad v_0 = 0.2, \\ \tau &= 0.337892, \quad Q = 10.5355, \quad \sigma = 3Q\tau c^2, \quad W = \frac{1}{Q\tau}. \end{aligned} \quad (73)$$

The asymptotic solution of system (69) can be presented in the form

$$u(r, t) = \varepsilon u_1(\varepsilon^\nu(r - ct); a\varepsilon^{\nu+1}t) = \varepsilon u_1(\varepsilon^\nu kr - (\varepsilon^\nu kc + \omega a\varepsilon^{\nu+1})t),$$

which corresponds to solution (70) with $K = \varepsilon^\nu k$, $W = \varepsilon^\nu kc + \omega a\varepsilon^{\nu+1}$.

$$\begin{aligned} \text{Here } a &= \frac{n+1}{2}, \quad k = \sqrt{\frac{\alpha}{\gamma}}, \quad \omega = 6\frac{\alpha^2}{\gamma}, \quad \beta^2 = 16\alpha\gamma, \quad \alpha = \frac{\varepsilon^{\nu-1}}{b(n+1)Q\tau} (bQ\tau^2c^2 - v_0^2), \\ \beta &= \frac{\varepsilon^{2\nu-1}}{b(n+1)} \left(c + \frac{\sigma}{Q\tau c} \right) (bQ\tau^2c^2 - v_0^2), \quad \gamma = \frac{\varepsilon^{3\nu-1}}{b(n+1)Q\tau c^2} (bQ\tau^2c^2 - v_0^2). \end{aligned}$$

That is, at $K = \varepsilon^\nu k$ and $W = kc\varepsilon^\nu + \omega a\varepsilon^{\nu+1}$ the asymptotic solutions are similar.

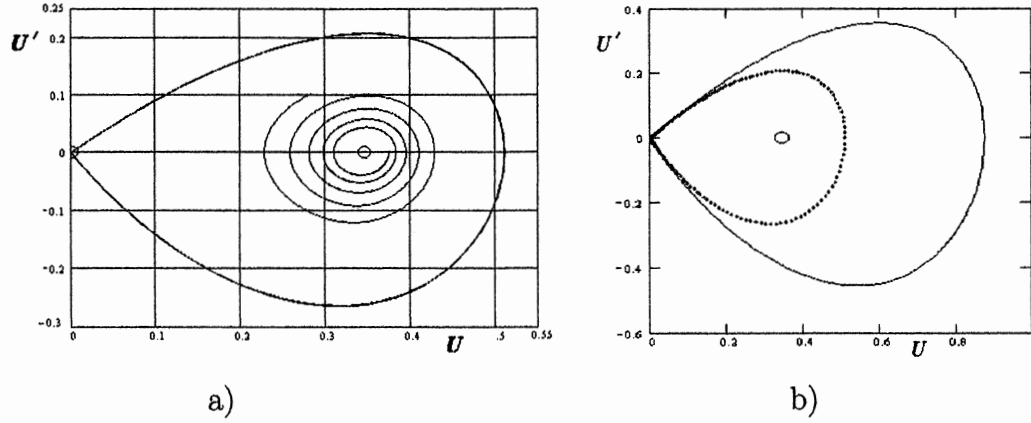


Fig. 1: The phase portrait (a) of system (72) and the comparison (b) of the numerical solution to system (72) (dotted line) with the asymptotic solution to system (69) (continuous line).

Thus, linearized dynamical system (72) describing the travelling wave solution of model (69) has the soliton solution (fig. 1a).

Given the numerical solutions shown, certain scales and values of parameters (73) exist, so that the numerical solution coincides with the asymptotic solution within the framework of the given approximation (fig.1b).

Mathematical modeling of spatial and temporal nonlocalities on nonlinear waves evolution is also presented in our articles [6–10] in more detail.

Bibliography

- [1] Danevych T.B., Danylenko V.A. Investigations of the influence of spatial nonlocality on the waves spreading in a relaxing medium // S.I.Subbotin institute of geophysics NAS of Ukraine, preprint. – Kyiv, 1994. – 22 p.
- [2] Danevych T.B., Danylenko V.A. Influence of spatial nonlocality on the waves spreading in a relaxing medium // Dop. NAS of Ukraine. – 1995. – Vol.11. – P. 83-86.
- [3] Kudryashov N.A. Exact solutions of N -order equations with Burgers-KdV nonlinearity // Matematicheskoe modelirovaniye, 1989. – Vol.1, No.6. – P. 57-65.
- [4] Danevych T.B., Danylenko V.A. Exact analytical solutions of nonlinear equations for dynamics of relaxing media with spatial and temporal nonlocalities. // Dop. NAS of Ukraine . - 2004. - No.3. - P. 110 - 114.
- [5] Danevych T.B., Danylenko V.A. Exact analytical solutions of nonlinear equations for dynamics of nonlocal media taking into account of temperature terms and the nonstationarity of generalized thermodynamical force. // Dop. NAS of Ukraine . - 2004. - No.4. - P. 106 - 112.
- [6] Makarenko A. S. New Differential Equation Model for Hydrodynamics with Memory Effects. // Reports on Mathematical Physics.- 2000. - Vol. 46, No. 1/2.

- [7] Makarenko A. S. Nonequilibrium Hydrodynamical Processes with Memory and Nonlocality: Mathematical Models and Their Solutions.- // Internationa J. Computing Anticipatory Systems, 2000.
- [8] Makarenko A. S. Model Equations and Structures Formation for the Media with Memory.// Ukrainian Journal of Physics, 2012.
- [9] Makarenko A. S. Complexity of Individual Objects Without Including Probability Concept. Asymmetry, Observer and Perception Problems.// Proceedings of Int. Conf. CASYS'2000 (Computer Anticipatory Systems), Liege, Belgium, August 2000, American Physical Society. Conference Proceedings. 2000.- Vol. 573, pp.201-215.
- [10] Makarenko A. S., Sagaydak L. Computation of Information Content of Data Analyzing their Symmetry.// IEEE Int. Workshop on Intelligent DATA Acquisition and Advanced Computing Systems Technology and Applications Bulgaria, 2005.

**PLANAR AND PROJECTIVE LINE GRAPHS ASSOCIATED
TO ZERO DIVISOR GRAPHS OF PARTIALLY ORDERED SETS**

A. Parsapour*

Department of Pure Mathematics
International Campus of Ferdowsi
University of Mashhad
P.O. Box 1159-91775, Mashhad, Iran
parsa216@gmail.com

Received June 10, 2015

Abstract

The zero divisor graph of a partially ordered set (poset, briefly) with the least element 0, which is denoted by $G^*(P)$, is an undirected graph with vertex set $P^* = P \setminus \{0\}$ and, for two distinct vertices x and y , x is adjacent to y in $G^*(P)$ if and only if $\{x, y\}^l = \{0\}$, where, for a subset S of P , the set S^l contains all elements $x \in P$ with $x \leq s$ for all $s \in S$. In this paper, we completely characterize all posets P with planar and projective line graphs associated to a zero divisor graph $G^*(P)$.

AMS Classification 05C10, 06A07

Keywords. Line graph, Partially ordered set, Planar graph, Projective graph, Zero divisor graph

*Corresponding author

1 Introduction

Recently, there has been considerable researches done on associating graphs with algebraic structures (cf. [2], [4], [11], [12] and [13]). The concept of the zero-divisor graph of a poset $(P, \leq)$, denoted by $G(P)$, was introduced by Liu and Xue [18]. For $x, y \in P$, let $\{x, y\}^l$ denote the set of lower bounds for the set $\{x, y\}$. $G(P)$ is a simple graph with vertex set P and two distinct vertices x and y in $G(P)$ are adjacent if and only if $\{x, y\}^l \neq \{0\}$. In [1], the authors omitted 0 from the vertex set of $G(P)$, and denoted the resulted graph by $G^*(P)$. Also in [1], the planarity of $G^*(P)$ is completely investigated. In this work, we characterize all posets P with planar and projective line graphs associated to zero divisor graphs $G^*(P)$, denoted by $\mathcal{L}(G^*(P))$.

We recall some definitions and notations on posets.

In a partially ordered set $(P, \leq)$ (poset, briefly) with a least element 0, an element a is called an *atom* if $a \neq 0$ and, for an element x in P , the relation $0 \leq x \leq a$ implies either $x = 0$ or $x = a$. We use the notation $A(P)$ for the set of atoms in P . Assume that S is a subset of P . Then an element x in P is a *lower bound* of S if $x \leq s$ for all $s \in S$. An *upper bound* is defined similarly. The set of all lower bounds of S is denoted by S^l and the set of all upper bounds of S by S^u , i.e.

$$S^l := \{x \in P \mid x \leq s, \text{ for all } s \in S\}$$

and

$$S^u := \{x \in P \mid s \leq x, \text{ for all } s \in S\}.$$

For basic facts concerning posets we refer the reader to [8].

Now, we recall some definitions and notations on graphs.

Let G be a simple graph with vertex-set $V(G)$ and edge-set $E(G)$. In a graph G , for two distinct vertices a and b in G , the notation $a - b$ means that a and b are adjacent. Also, the *degree* of a vertex x , denoted by $\deg(x)$, is the number of edges incident to x , and an *isolated vertex* is a vertex with zero degree. A graph with no edges (but at least one vertex) is called an *empty graph*. The graph with no vertices and no edges is the *null graph*. For a positive integer r , an *r -partite graph* is one whose vertex-set can be partitioned into r subsets, so that no edge has both ends in any one subset.

A *complete r -partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. For positive integers m and n , the graph K_n is a complete graph with n vertices and the graph $K_{m,n}$ is a complete bipartite graph, with parts of sizes m and n . A complete bipartite graph $K_{1,n}$ is called *star* (see [5] and [10]). A graph G is said to be *contracted* to a graph H if there exists a sequence of elementary contractions which transforms G into H , where an *elementary contraction* consists of deletion of a vertex or an edge or the identification of two adjacent vertices. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. The line graph of a graph G is the graph $\mathcal{L}(G)$ with the edges of G as its vertices, and two edges of G are adjacent in $\mathcal{L}(G)$ if and only if they are incident in G . In this work, we denote $w_{i,j}$ for the vertex $[v_i, v_j] \in \mathcal{L}(G)$, where v_i and v_j are adjacent vertices in G .

In Section 3 of this work, we study the planarity of the line graph associated to zero divisor graph of posets. In Section 4, we characterize all posets P that, the graph $\mathcal{L}(G^*(P))$ is projective.

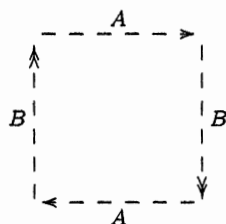
2 Basic definitions and properties

Recall that a graph is said to be *planar* if it can be drawn in the plane, such that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [5, p. 153]). In [17], the author characterized the planarity of a line graph $\mathcal{L}(G)$ by using the planarity of G and its vertex degrees.

Theorem 2.1. [17, Lemma 2.6] *A non-empty graph G has a planar line graph $\mathcal{L}(G)$ if and only if*

- (i) G is planar,
- (ii) $\Delta(G) \leq 4$, and
- (iii) if $\deg(v) = 4$, then v is a cut-vertex in the graph G .

By a *surface*, we mean a connected compact 2-dimensional real manifold without boundary, that is a connected topological space such that each point has a neighbourhood homeomorphic to an open disc. It is well-known that every compact surface is homeomorphic to a sphere, or to a connected sum of g tori, or to a connected sum of k projective planes (see [14, Theorem 5.1]). This number k is called the *crosscap number* of the surface. Also, we denote the crosscap number of a graph G by $\bar{\gamma}(G)$. One easy observation is that $\bar{\gamma}(H) \leq \bar{\gamma}(G)$, for any subgraph H of G . The projective plane can be thought of as a sphere with one crosscap. This means that the crosscap number of projective plane is 1.



The canonical representation of a projective plane

A graph G is *embeddable* in a surface S if the vertices of G are distinct points in S and every edge of G is a simple arc in S connecting the two vertices which is joined in G . A *projective graph* is a graph that can be embedded in projective plane. If G can not be embedded in S , then G has at least two edges intersecting at a point which is not a vertex of G . We say a graph G is *irreducible* for a surface S if G does not embed in S , but any proper subgraph of G embeds in S . The set of 103 irreducible graphs for the projective plane has been found by H. Glover, J.P. Huneke, and C.S. Wang in [9], and D. Archdeacon in [3] proved that this list is complete. This list also has been checked by Myrvold and Roth in [15]. Hence a graph embeds in the projective plane, which is called a *projective graph*, if and only if it contains no subdivision of 103 graphs in [9]. Also, a complete graph K_n is projective if $n = 5$ or 6 , and the only projective complete bipartite graphs are $K_{3,3}$ and $K_{3,4}$ (see [6] or [16]). Note that a planar graph is not considered as a projective graph.

We state the following notation, which is needed in the rest of this paper.

Notation 2.2. To simplify notations, we set $\mathfrak{p} := \{p\}^u$, where $p \in A(P)$. Suppose that $A(P) = \{p_1, p_2, \dots, p_n\}$, where $n > 1$, and $1 \leq i_1 < i_2 < \dots < i_k \leq n$. The notation $P_{i_1 i_2 \dots i_k}$ stands for the following set

$$\{x \in P; x \in \bigcap_{s=1}^k \mathfrak{p}_{i_s} \setminus \bigcup_{j \neq i_1, i_2, \dots, i_k} \mathfrak{p}_j\}.$$

Note that no two distinct elements in $P_{i_1 i_2 \dots i_k}$ are adjacent in graph $G^*(P)$. Also if the index sets $\{i_1, i_2, \dots, i_k\}$ and $\{j_1, j_2, \dots, j_{k'}\}$ of $P_{i_1 i_2 \dots i_k}$ and $P_{j_1 j_2 \dots j_{k'}}$, respectively, are distinct, then one can easily check that $P_{i_1 i_2 \dots i_k} \cap P_{j_1 j_2 \dots j_{k'}} = \emptyset$. Moreover, $P \setminus \{0\} = \bigcup_{k=1, 1 \leq i_1 < i_2 < \dots < i_k \leq n}^n P_{i_1 i_2 \dots i_k}$.

Note that $P_{12 \dots n}$ is the set of isolated points in $G^*(P)$ and also $a_i \in P_i$, for all $1 \leq i \leq n$.

3 On the planarity of $\mathfrak{L}(G^*(P))$

In this section, we explore the planarity of the line graph associated to the graph $G^*(P)$. Note that if $|A(P)| = 1$, then $G^*(P)$ has no edges at all. It means that $G^*(P)$ is an empty graph, and hence $\mathfrak{L}(G^*(P))$ is the null graph.

Now, we state the following lemma, which is useful.

Lemma 3.1. If $\mathfrak{L}(G^*(P))$ is planar, then the size of $A(P)$ is at most four.

Proof. Assume to the contrary that $|A(P)| \geq 5$. Since the induced subgraph of $G^*(P)$ on the vertices $v_1 \in P_1$, $v_2 \in P_2$, $v_3 \in P_3$, $v_4 \in P_4$ and $v_5 \in P_5$ contains a copy of K_5 , we can find a subgraph of $\mathfrak{L}(G^*(P))$ isomorphic to a subdivision of $K_{3,3}$ (see Figure 1).

Therefore it is not a planar graph, which is a contradiction. $\square$

By Lemma 3.1, we must study the cases that $|A(P)|$ is equal to 1, 2, 3 and 4. If $|A(P)| = 1$, then the graph $G^*(P)$ is empty, and so $\mathfrak{L}(G^*(P))$ is the null graph. In the following theorem, we state the necessary and sufficient condition for the planarity of the graph $\mathfrak{L}(G^*(P))$, when $|A(P)| = 2$.

Theorem 3.2. Suppose that $|A(P)| = 2$. Then $\mathfrak{L}(G^*(P))$ is a planar graph if and only if $|\bigcup_{j=1}^2 P_j| \leq 5$.

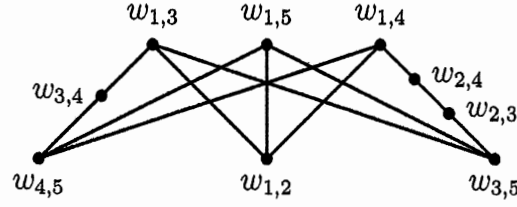


Figure 1:

Proof. First, assume that $\mathfrak{L}(G^*(P))$ is planar and assume on the contrary that $|\bigcup_{j=1}^2 P_j| \geq 6$. We know that as $|A(P)| = 2$, the graph $G^*(P)$ is a complete bipartite graph. If $G^*(P)$ is a star graph, then the graph $\mathfrak{L}(G^*(P))$ contains a subgraph isomorphic to K_5 , which is not planar. Otherwise, $G^*(P)$ is not a star graph. Then it contains a subgraph isomorphic to $K_{2,4}$ or $K_{3,3}$. In these two cases, $\mathfrak{L}(G^*(P))$ contains a subdivision of $K_{3,3}$. Hence $\mathfrak{L}(G^*(P))$ is not planar, which is a contradiction.

Conversely, suppose that $|\bigcup_{j=1}^2 P_j| \leq 5$. If $|\bigcup_{j=1}^2 P_j| = 2$, then $\mathfrak{L}(G^*(P))$ is isomorphic to $\mathfrak{L}(K_2)$, which is an empty graph with one vertex. Also if $|\bigcup_{j=1}^2 P_j| = 3$, then $\mathfrak{L}(G^*(P)) \cong \mathfrak{L}(K_{1,2}) \cong K_2$. In addition, if $|\bigcup_{j=1}^2 P_j| = 4$, then $G^*(P)$ is isomorphic to $K_{1,3}$ or $K_{2,2}$. Hence $\mathfrak{L}(G^*(P))$ is isomorphic to K_3 and $K_{2,2}$, respectively. Finally, assume that $|\bigcup_{j=1}^2 P_j| = 5$. If $G^*(P)$ is a star graph, then $\mathfrak{L}(G^*(P)) \cong K_4$. Otherwise, the graph $G^*(P)$ is isomorphic to $K_{2,3}$ with vertices $v_1, v_2, v_3 \in P_1$, $v_4, v_5 \in P_2$. In this case, the graph $\mathfrak{L}(G^*(P))$ is pictured in Figure 2. In all of the above situations,

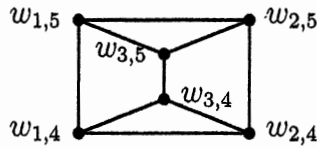


Figure 2:

$\mathfrak{L}(G^*(P))$ is a planar graph. □

Now, we investigate the planarity of $\mathfrak{L}(G^*(P))$, when $|A(P)| = 3$. Suppose that $|\bigcup_{j=1}^3 P_j| \geq 5$. It is easy to see that $G^*(P)$ contains a subgraph

isomorphic to a complete 3-partite graph $K_{3,1,1}$ or $K_{2,2,1}$. Therefore the graph $\mathfrak{L}(G^*(P))$ contains a subdivision of $K_{3,3}$ and subdivision of K_5 , respectively. Hence it is not planar, and so we proved the following result.

Lemma 3.3. *If $\mathfrak{L}(G^*(P))$ is planar, then $|\bigcup_{j=1}^3 P_j| \leq 4$.*

Theorem 3.4. *Suppose that $|A(P)| = 3$. Then $\mathfrak{L}(G^*(P))$ is a planar graph if and only if one of the following conditions holds:*

- (i) $|\bigcup_{j=1}^3 P_j| = 3$ and $|P_{ij}| \leq 2$, for $1 \leq i, j \leq 3$.
- (ii) $|\bigcup_{j=1}^3 P_j| = 4$ and $|P_{ij}| \leq 1$, for $1 \leq i, j \leq 3$.

Proof. First, assume that one of the conditions (i) or (ii) holds. Suppose that $|\bigcup_{j=1}^3 P_j| = 3$ and $|P_{23}| = |P_{12}| = |P_{13}| = 2$. The graph $G^*(P)$ with vertices $v_1 \in P_1$, $v_2 \in P_2$, $v_3 \in P_3$, $v_4, v_5 \in P_{23}$, $v_6, v_7 \in P_{12}$ and $v_8, v_9 \in P_{13}$ is pictured in Figure 3.

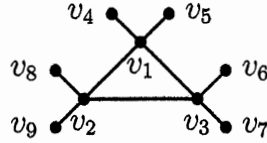


Figure 3:

Hence the graph $\mathfrak{L}(G^*(P))$ is pictured in Figure 4, which is planar.

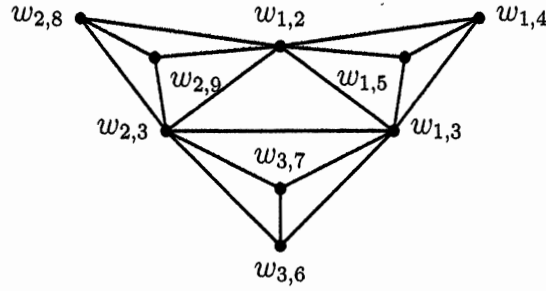


Figure 4:

Now, suppose that $|\bigcup_{j=1}^3 P_j| = 4$, $|P_1| = 2$ and $|P_{23}| = |P_{13}| = |P_{12}| = 1$. The graph $G^*(P)$ with vertices $v_1, v_2 \in P_1$, $v_3 \in P_2$, $v_4 \in P_3$, $v_5 \in P_{23}$, $v_6 \in P_{13}$, $v_7 \in P_{12}$ is pictured in Figure 5 and $\mathfrak{L}(G^*(P))$ is pictured in Figure 6, which is a planar graph.

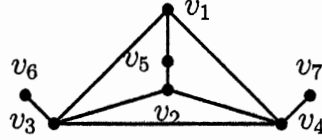


Figure 5:

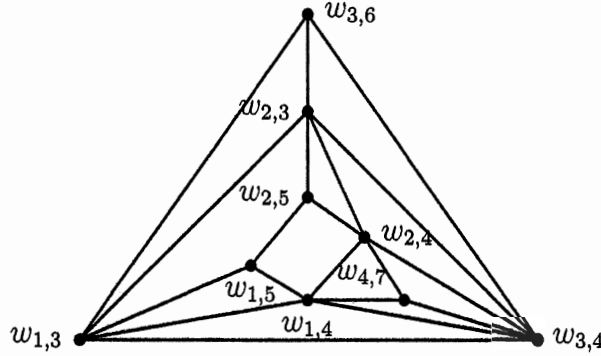


Figure 6:

Conversely, suppose that $\mathfrak{L}(G^*(P))$ is a planar graph. By Lemma 3.3, $|\bigcup_{j=1}^3 P_j| \leq 4$. Hence we have the following cases.

Case 1. $|\bigcup_{j=1}^3 P_j| = 3$. If P_{12} , P_{13} or P_{23} has at least three elements, then there exist at least a vertex of degree five in the graph $G^*(P)$. Hence the graph $\mathfrak{L}(G^*(P))$ contains a subgraph isomorphic to K_5 , and so it is not planar, which is a contradiction.

Case 2. $|\bigcup_{j=1}^3 P_j| = 4$. Without loss of generality, we may assume that $|P_1| = 2$. If P_{12} or P_{13} has at least two elements, then there exist at least a vertex of degree five in the graph $G^*(P)$. Hence the graph $\mathfrak{L}(G^*(P))$ contains a copy of K_5 , and so it is not planar, which is a contradiction. In

addition, if P_{23} has at least two elements, then $G^*(P)$ contains a subgraph isomorphic to $K_{2,4}$. Therefore $G^*(P)$ has a vertex of degree four which is not a cut-vertex. By Theorem 2.1, $\mathfrak{L}(G^*(P))$ is not a planar graph, which is a contradiction. $\square$

Finally, in order to complete the study of the planarity of $\mathfrak{L}(G^*(P))$, we assume that $|A(P)| = 4$.

Lemma 3.5. *If $\mathfrak{L}(G^*(P))$ is planar, then $|\bigcup_{j=1}^4 P_j| = 4$.*

Proof. Suppose on the contrary that $|\bigcup_{j=1}^4 P_j| \geq 5$. Then the graph $G^*(P)$ has a vertex of degree four which is not a cut-vertex. Hence, by Theorem 2.1, $\mathfrak{L}(G^*(P))$ is not a planar graph, which is a contradiction. $\square$

Theorem 3.6. *Suppose that $|A(P)| = 4$. Then $\mathfrak{L}(G^*(P))$ is a planar graph if and only if $P_{ij} = \emptyset$ whenever $|P_{ijk}| \leq 1$, for all $i, j, k \in \{1, 2, 3, 4\}$.*

Proof. First, assume that the graph $\mathfrak{L}(G^*(P))$ is planar. By Lemma 3.5, we have $|\bigcup_{j=1}^4 P_j| = 4$. If there exist at least one element in P_{ij} , for $i, j \in \{1, 2, 3, 4\}$, then one can easily check that the graph $\mathfrak{L}(G^*(P))$ contains a subdivision of $K_{3,3}$, which is not planar. Also if one of the sets P_{ijk} has at least two elements, for $i, j, k \in \{1, 2, 3, 4\}$, then the graph $G^*(P)$ has a vertex of degree five. Hence the graph $\mathfrak{L}(G^*(P))$ contains a copy of K_5 , which is impossible.

Conversely, suppose that $P_{12} = P_{13} = P_{23} = \emptyset$ and $|P_{123}| = |P_{124}| = |P_{134}| = |P_{234}| = 1$. The graph $G^*(P)$ with vertices $v_1 \in P_1$, $v_2 \in P_2$, $v_3 \in P_3$, $v_4 \in P_4$, $v_5 \in P_{123}$, $v_6 \in P_{124}$, $v_7 \in P_{134}$ and $v_8 \in P_{234}$ is pictured in Figure 7.

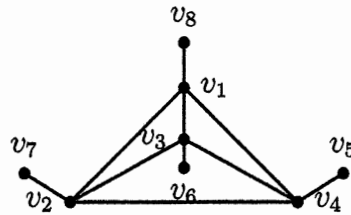


Figure 7:

Hence $\mathfrak{L}(G^*(P))$ is pictured in Figure 8, which is a planar graph. $\square$

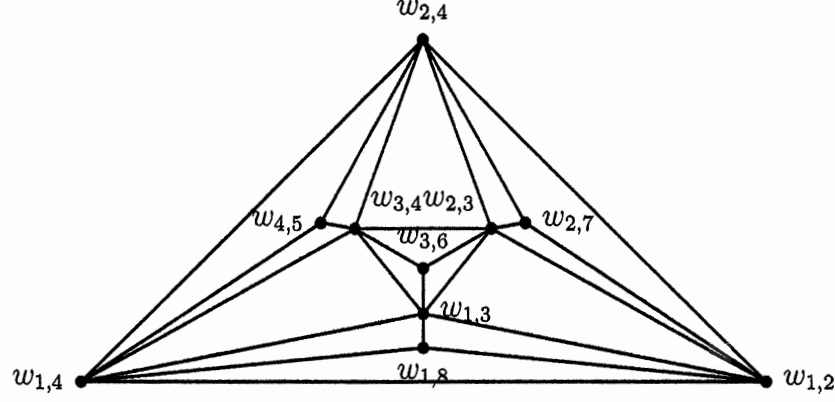


Figure 8:

Therefore in the case that $P_{ij} = \emptyset$ whenever $|P_{ijk}| \leq 1$, for all $i, j, k \in \{1, 2, 3, 4\}$, we have $\mathfrak{L}(G^*(P))$ is planar.

4 On the Projectivity of $\mathfrak{L}(G^*(P))$

In this section, we investigate the projectivity of the line graph of the graph $G^*(P)$. We begin this section with the following lemma.

Lemma 4.1. *If $\mathfrak{L}(G^*(P))$ is projective, then the size of $A(P)$ is at most four.*

Proof. Assume to the contrary that $|A(P)| \geq 5$. Then the graph $G^*(P)$ contains a copy of K_5 with vertices $v_1 \in P_1$, $v_2 \in P_2$, $v_3 \in P_3$, $v_4 \in P_4$ and $v_5 \in P_5$. So the contraction of the graph $\mathfrak{L}(G^*(P))$ contains a copy of E_{20} , one of the listed graphs in [9] (see Figure 9). Therefore $\mathfrak{L}(G^*(P))$ is not a projective graph, which is a contradiction. $\square$

By Lemma 4.1, it is sufficient to investigate the projectivity of the graph $\mathfrak{L}(G^*(P))$ in the cases that the size of $A(P)$ is 2, 3 or 4.

Theorem 4.2. *Suppose that $|A(P)| = 2$. Then $\mathfrak{L}(G^*(P))$ is a projective graph if and only if one of the following conditions holds:*

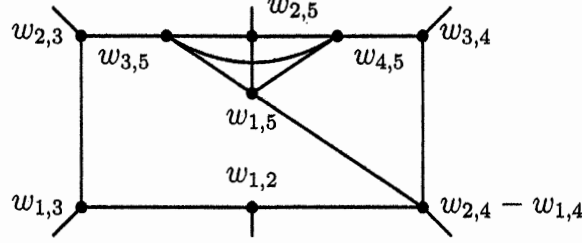


Figure 9:

(i) $|\bigcup_{j=1}^2 P_j| = 6$, and $|P_i| = 1$, for some unique $i \in \{1, 2\}$ or $|P_i| = |P_j| = 3$, for $i, j \in \{1, 2\}$.

(ii) $|\bigcup_{j=1}^2 P_j| = 7$ and $|P_i| = 1$, for some unique $i \in \{1, 2\}$.

Proof. First, assume that the graph $\mathfrak{L}(G^*(P))$ is projective and on the contrary, $|\bigcup_{j=1}^2 P_j| \leq 5$. Then, by Theorem 3.2, the graph $\mathfrak{L}(G^*(P))$ is planar, which is not projective. Now, assume that $|\bigcup_{j=1}^2 P_j| = 6$ and $G^*(P) \cong K_{2,4}$. By [7, Example 2.14], $\bar{\gamma}(\mathfrak{L}(K_{2,4})) = 2$, and so the graph $\mathfrak{L}(G^*(P))$ is not projective. Hence, if $|\bigcup_{j=1}^2 P_j| = 6$, then the statement (i) holds. Now, suppose that $|\bigcup_{j=1}^2 P_j| = 7$. If $G^*(P)$ is not a star graph, then it is isomorphic to $K_{2,5}$ or $K_{3,4}$. By [7, Corollary 2.11], $\bar{\gamma}(\mathfrak{L}(K_{2,5})) = 2$ and, by [7, Example 2.14], $\bar{\gamma}(\mathfrak{L}(K_{3,4})) = 2$. So if $|\bigcup_{j=1}^2 P_j| = 7$, then the statement (ii) holds. Finally, we may assume that $|\bigcup_{j=1}^2 P_j| \geq 8$. If $G^*(P)$ is a star graph, then the graph $\mathfrak{L}(G^*(P))$ contains a subgraph isomorphic to K_7 , which is not projective. Otherwise, $G^*(P)$ is not a star graph. Then it contains a subgraph isomorphic to $K_{2,6}$, $K_{3,5}$ or $K_{4,4}$. In these cases, $G^*(P)$ contains a copy of $K_{2,4}$. Clearly, $\bar{\gamma}(\mathfrak{L}(G^*(P))) \geq \bar{\gamma}(\mathfrak{L}(K_{2,4}))$, and we have $\bar{\gamma}(\mathfrak{L}(K_{2,4})) = 2$. It means that the graph $\mathfrak{L}(G^*(P))$ is not projective. Therefore, if $\mathfrak{L}(G^*(P))$ is projective, then one of the conditions (i) or (ii) holds.

Conversely, suppose that $|\bigcup_{j=1}^2 P_j| = 6$, and the graph $G^*(P)$ is a star graph. Then $\mathfrak{L}(G^*(P)) \cong K_5$, and so it is a projective graph. Now, suppose that $G^*(P) \cong K_{3,3}$. By [7, Example 2.12], $\bar{\gamma}(\mathfrak{L}(K_{3,3})) = 1$, and so the graph $\mathfrak{L}(G^*(P))$ is projective. Finally, suppose that $|\bigcup_{j=1}^2 P_j| = 7$, and the graph $G^*(P)$ is a star graph. Then $\mathfrak{L}(G^*(P)) \cong K_6$, and so it is a projective graph. $\square$

Now, we investigate the projectivity of $\mathfrak{L}(G^*(P))$, when $|A(P)| = 3$. Suppose that $|\bigcup_{j=1}^3 P_j| \geq 6$. Then the graph $G^*(P)$ contains a subgraph isomorphic to $K_{4,1,1}$, $K_{3,2,1}$ or $K_{2,2,2}$. If $G^*(P)$ contains a subgraph isomorphic to $K_{4,1,1}$, then one can easily find a copy of A_1 , one of the listed graphs in [9], in the graph $\mathfrak{L}(G^*(P))$, which is not projective. Also if $G^*(P)$ contains a subgraph isomorphic to $K_{3,2,1}$, then one can easily find a copy of E_{20} , one of the listed graphs in [9], in the contraction of $\mathfrak{L}(G^*(P))$, which is not projective. Now, if $G^*(P)$ contains a subgraph isomorphic to $K_{2,2,2}$, then the contraction of $\mathfrak{L}(G^*(P))$ contains a copy of E_3 , one of the listed graphs in [9], which is not projective. Therefore $\mathfrak{L}(G^*(P))$ is not a projective graph.

As a consequence of the above discussion, we state the following lemma.

Lemma 4.3. *If $\mathfrak{L}(G^*(P))$ is projective, then $|\bigcup_{j=1}^3 P_j| \leq 5$.*

Theorem 4.4. *Suppose that $|A(P)| = 3$. Then $\mathfrak{L}(G^*(P))$ is a projective graph if and only if one of the following conditions holds:*

- (i) $|\bigcup_{j=1}^3 P_j| = 3$, there exist unique i and j , with $1 \leq i, j \leq 3$, such that $3 \leq |P_{ij}| \leq 4$ whenever $|P_{kk'}| \leq 2$, for $k \in \{i, j\}$ and $\{k'\} = \{1, 2, 3\} \setminus \{i, j\}$.
- (ii) $|\bigcup_{j=1}^3 P_j| = 4$, there exists unique i , with $1 \leq i \leq 3$, such that $|P_i| = 2$, and for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, if $2 \leq |P_{ij}| \leq 3$, then $|P_{ik}| \leq 1$ whenever $P_{jk} = \emptyset$.
- (iii) $|\bigcup_{j=1}^3 P_j| = 5$,
 - (a) there exists unique i , with $1 \leq i \leq 3$, such that $|P_i| = 3$, and for all $1 \leq j, k \leq 3$, $P_{jk} = \emptyset$.
 - (b) there exists unique i , with $1 \leq i \leq 3$, such that $|P_i| = 1$, and for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, $|P_{jk}| \leq 1$ whenever $P_{ij} = P_{ik} = \emptyset$.

Proof. First we assume that $\mathfrak{L}(G^*(P))$ is a projective graph. By Lemma 4.3, $|\bigcup_{j=1}^3 P_j| \leq 5$. We have the following cases.

Case 1. $|\bigcup_{j=1}^3 P_j| = 3$. In this case, if $|P_{ij}| \leq 2$, for all $i, j \in \{1, 2, 3\}$, then by Theorem 3.4, the graph $\mathfrak{L}(G^*(P))$ is planar, which is not projective. Also if without loss of generality, $|P_{12}|, |P_{13}| \in \{3, 4\}$, then one can easily

check that the graph $\mathfrak{L}(G^*(P))$ contains a copy of A_1 , one of the listed graphs in [9], which is not projective. In addition, if without loss of generality, P_{12} , P_{13} or P_{23} has at least five elements, then the graph $\mathfrak{L}(G^*(P))$ contains a subgraph isomorphic to K_7 , which is not projective. Therefore, for the projectivity of $\mathfrak{L}(G^*(P))$, it is necessary that there exist unique i and j , with $1 \leq i, j \leq 3$, such that $3 \leq |P_{ij}| \leq 4$ whenever $|P_{kk'}| \leq 2$, for $k \in \{i, j\}$ and $\{k'\} = \{1, 2, 3\} \setminus \{i, j\}$.

Case 2. $|\bigcup_{j=1}^3 P_j| = 4$. In this case, if $|P_{ij}| \leq 1$, for all $i, j \in \{1, 2, 3\}$, then by Theorem 3.4, the graph $\mathfrak{L}(G^*(P))$ is planar, which is not projective. Now, suppose that there exists unique P_i , with $1 \leq i \leq 3$, say P_1 , such that $|P_1| = 2$. If $|P_{23}| \geq 2$, then $G^*(P)$ contains a copy of $K_{2,4}$. Clearly, $\overline{\gamma}(\mathfrak{L}(G^*(P))) \geq \overline{\gamma}(\mathfrak{L}(K_{2,4}))$, and we have $\overline{\gamma}(\mathfrak{L}(K_{2,4})) = 2$. It implies that the graph $\mathfrak{L}(G^*(P))$ is not projective. Also if $|P_{12}| = 2$ and $|P_{23}| = 1$, then the contraction of $\mathfrak{L}(G^*(P))$ contains B_1 , one of the listed graphs in [9], which is not projective. Now, we may assume that $P_{23} = \emptyset$. If P_{12} or P_{13} has at least four elements, then the graph $\mathfrak{L}(G^*(P))$ contains a subgraph isomorphic to K_7 , which is not projective. Also if $|P_{12}| = |P_{13}| = 2$, then the graph $\mathfrak{L}(G^*(P))$ contains a copy of A_1 , one of the listed graphs in [9], which is not projective. Therefore, for the projectivity of $\mathfrak{L}(G^*(P))$, it is necessary that $2 \leq |P_{ij}| \leq 3$, $|P_{ik}| \leq 1$ whenever $P_{jk} = \emptyset$, for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, when $|P_i| = 2$.

Case 3. $|\bigcup_{j=1}^3 P_j| = 5$. Suppose that $|P_1| = 3$. If P_{12} or P_{13} has at least one element, then the graph $\mathfrak{L}(G^*(P))$ contains a copy of D_{17} , one of the listed graphs in [9], which is not projective. Also if P_{23} has at least one element, then the contraction of $\mathfrak{L}(G^*(P))$ contains a copy of E_{20} , one of the listed graphs in [9], which is not a projective graph. Therefore, for the projectivity of $\mathfrak{L}(G^*(P))$, it is necessary that $P_{12} = P_{13} = P_{23} = \emptyset$, when $|P_1| = 3$. On the other hand, suppose that there exists unique P_i , with $1 \leq i \leq 3$, say P_1 , such that $|P_1| = 1$. If P_{12} or P_{13} has at least one element, then the contraction of $\mathfrak{L}(G^*(P))$ contains a copy of E_{20} , one of the listed graphs in [9], which is not a projective graph. Also if $|P_{23}| \geq 2$, then the contraction of $\mathfrak{L}(G^*(P))$ contains a copy of D_{17} , one of the listed graphs in [9], which is not a projective graph. Therefore, for the projectivity of $\mathfrak{L}(G^*(P))$, it is necessary that $P_{12} = P_{13} = \emptyset$ whenever $|P_{23}| \leq 1$, when $|P_1| = 1$.

Conversely, if one of the statements (i), (ii) or (iii) holds, then we show that $\mathfrak{L}(G^*(P))$ is a projective graph.

First suppose that $|\bigcup_{j=1}^3 P_j| = 3$. If $|P_{12}| = |P_{13}| = 2$ and $|P_{23}| = 4$, then the graph $G^*(P)$ is pictured in Figure 10, which is planar and the graph $\mathfrak{L}(G^*(P))$ is pictured in Figure 11, which is projective. We have $v_1 \in P_1$, $v_2 \in P_2$, $v_3 \in P_3$, $v_4, v_5 \in P_{12}$, $v_6, v_7 \in P_{13}$ and $v_8, v_9, v_{10}, v_{11} \in P_{23}$.

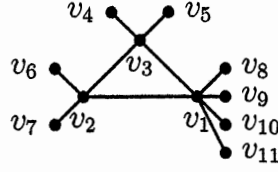


Figure 10:

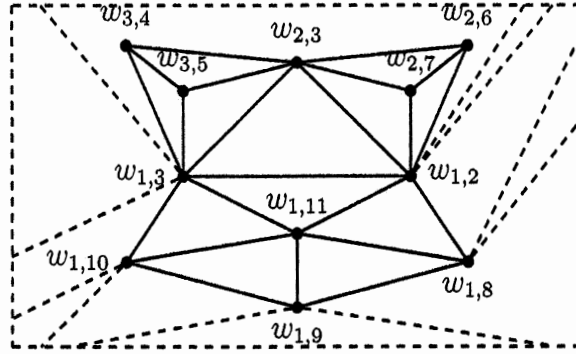


Figure 11:

Now, suppose that $|\bigcup_{j=1}^3 P_j| = 4$ and $|P_1| = 2$. If $|P_{12}| = 1$, $|P_{13}| = 3$ and $P_{23} = \emptyset$, then the graph $G^*(P)$ with vertices $v_1, v_2 \in P_1$, $v_3 \in P_2$, $v_4 \in P_3$, $v_5 \in P_{12}$ and $v_6, v_7, v_8 \in P_{13}$ is planar and the graph $\mathfrak{L}(G^*(P))$ is projective (see Figure 12).

Finally, suppose that $|\bigcup_{j=1}^3 P_j| = 5$ and consider the following cases.

Case 1. There exists unique S_i , with $1 \leq i \leq 3$, say P_1 , such that $|P_1| = 3$, and also $P_{12} = P_{13} = P_{23} = \emptyset$. Then the graph $G^*(P)$ with

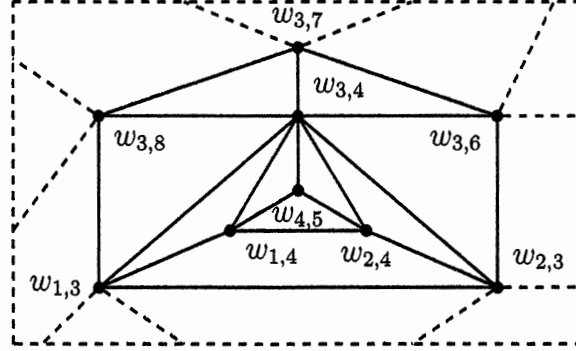


Figure 12:

vertices $v_1, v_2, v_3 \in P_1$, $v_4 \in P_2$ and $v_5 \in P_3$ is planar. As observed, in Figure 13, the graph $\mathcal{L}(G^*(P))$ is projective.

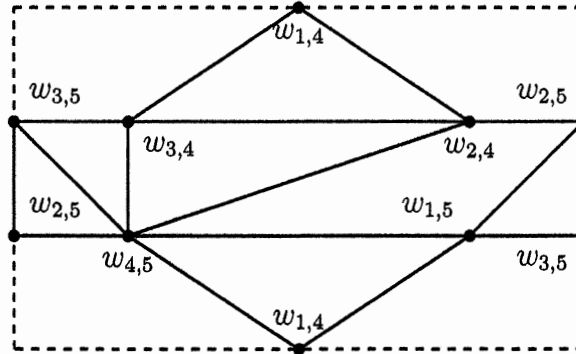


Figure 13:

Case 2. There exists unique P_i , with $1 \leq i \leq 3$, say P_1 , such that $|P_1| = 1$, also $P_{12} = P_{13} = \emptyset$ and $|P_{23}| = 1$. Then the graph $G^*(P)$ with vertices $v_1 \in P_1$, $v_2, v_3 \in P_2$, $v_4, v_5 \in P_3$ and $v_6 \in P_{23}$ is planar, and so $\mathcal{L}(G^*(P))$ is pictured in Figure 14, is a projective graph. $\square$

For the reminder of this section, suppose that $|A(P)| = 4$. Firstly, consider that $|\bigcup_{j=1}^4 P_j| = 5$ and $|P_1| = 2$. Then $\mathcal{L}(G^*(P))$ contains a subgraph isomorphic to E_{20} , one of the listed graphs in [9], which is not a projective graph. Therefore we have the following lemma.

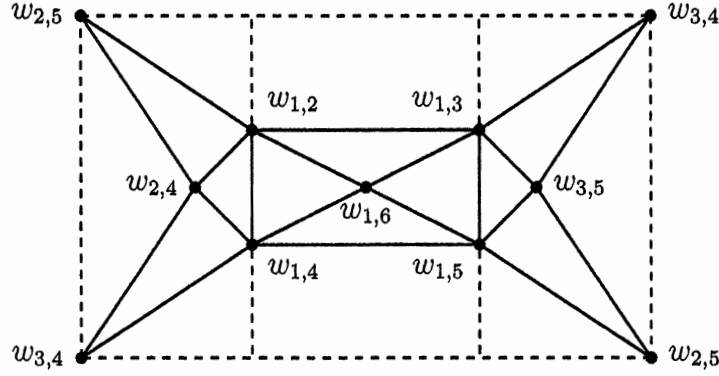


Figure 14:

Lemma 4.5. *If $\mathfrak{L}(G^*(P))$ is projective, then $|\bigcup_{j=1}^4 P_j| = 4$.*

In the sequel, suppose that $|\bigcup_{j=1}^4 P_j| = 4$. We have the following situations.

- (i) There exist i, j , with $1 \leq i, j \leq 4$, such that $|P_{ij}| \geq 2$. Then $\mathfrak{L}(G^*(P))$ contains a copy of A_1 , one of the listed graphs in [9], which is not a projective graph.
- (ii) There exist $i \neq i', j \neq j'$, with $1 \leq i, i', j, j' \leq 4$, such that $|P_{ij}| = |P_{i'j'}| = 1$. Then the contraction $\mathfrak{L}(G^*(P))$ contains a copy of D_{17} , one of the listed graphs in [9], which is not a projective graph.
- (iii) There exist $i \neq i', j$, with $1 \leq i, i', j \leq 4$, such that $|P_{ij}| = |P_{i'j}| = 1$. Then $\mathfrak{L}(G^*(P))$ contains a copy of D_{17} , one of the listed graphs in [9], which is not a projective graph.
- (iv) For all $1 \leq i, j, k \leq 4$, $|P_{ijk}| \leq 1$ whenever $P_{ij} = \emptyset$. Then, by Theorem 3.6, the graph $\mathfrak{L}(G^*(P))$ is planar, which is not projective.
- (v) There exist i, j, k , with $1 \leq i, j, k \leq 4$, such that $|P_{ijk}| \geq 4$. Then $\mathfrak{L}(G^*(P))$ contains a copy of K_7 , which is not projective.

- (vi) There exist unique $i \neq i', j, k$, with $1 \leq i, i', j, k \leq 4$, such that $2 \leq |S_{ijk}| \leq 3$ whenever $|P_{i'ij}| = |P_{i'ik}| = |P_{i'jk}| = 1$. Then the graph $G^*(P)$, with vertices $v_1 \in P_1, v_2 \in P_2, v_3 \in P_3, v_4 \in P_4, v_5, v_6, v_7 \in P_{123}, v_8 \in P_{124}, v_9 \in P_{134}$ and $v_{10} \in P_{234}$, is planar. Therefore the graph $\mathfrak{L}(G^*(P))$, which is pictured in Figure 15, is projective.

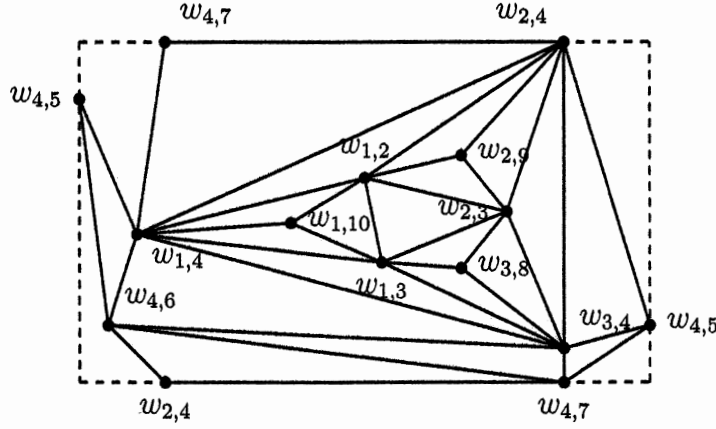


Figure 15:

- (vii) There exist $i \neq i', j, k$, with $1 \leq i, i', j, k \leq 4$, such that $|P_{ijk}| = |P_{i'jk}| = 2$. Then $\mathfrak{L}(G^*(P))$ contains a copy of A_1 , one of the listed graphs in [9], which is not a projective graph.
- (viii) There exist $i, j \neq j', k \neq k'$, with $1 \leq i, j, j', k, k' \leq 4$, $|P_{ij}| = 1$ whenever $|P_{ij'k'}| = 2$. Then the contraction of $\mathfrak{L}(G^*(P))$ contains a copy of B_1 , one of the listed graphs in [9], which is not a projective graph.
- (ix) There exist i, j, k , with $1 \leq i, j, k \leq 4$, $|P_{ij}| = |P_{ijk}| = 1$. Then $\mathfrak{L}(G^*(P))$ contains a copy of E_{19} , one of the listed graphs in [9], which is not a projective graph.
- (x) There exist unique i, i', j, j' , with $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, such that $|P_{ij}| = |P_{ii'j'}| = |P_{jj'j'}| = 1$. Then the graph $G^*(P)$, with vertices

$v_1 \in P_1, v_2 \in P_2, v_3 \in P_3, v_4 \in P_4, v_5 \in P_{12}, v_6 \in P_{134}$ and $v_7 \in P_{234}$, is planar. Therefore the graph $\mathfrak{L}(G^*(P))$, which is pictured in Figure 16, is projective.

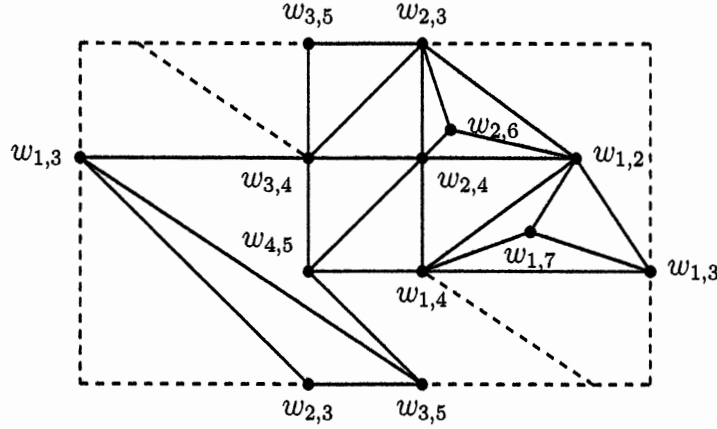


Figure 16:

As a consequence of the above discussion and Lemma 4.5, we state the necessary and sufficient conditions for the projectivity of the graph $\mathfrak{L}(G^*(P))$, when $|A(P)| = 4$.

Theorem 4.6. *Suppose that $|A(P)| = 4$. Then $\mathfrak{L}(G^*(P))$ is a projective graph if and only if $|\bigcup_{j=1}^4 P_j| = 4$ and one of the following conditions holds:*

- (i) *There exist unique $i \neq i', j, k$, with $1 \leq i, i', j, k \leq 4$, such that $2 \leq |P_{ijk}| \leq 3$ whenever $|P_{i'ij}| = |P_{i'ik}| = |P_{i'jk}| = 1$.*
- (ii) *There exist unique i, i', j, j' , with $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, such that $|P_{ij}| = |P_{ii'j'}| = |P_{jj'j'}| = 1$.*

Acknowledgments. The author would like to thank Prof. Kazem Khashyarmanesh and Dr. Mojgan Afkhami for their valuable comments and guidance to improve the quality of this paper.

References

- [1] M. AFKHAMİ, Z. BARATI, K. KHASHYARMANESH, *Planar zero divisor graphs of partially ordered sets*, Acta Math. Hungar. **137** (2012) 27-35.
- [2] D.F. ANDERSON, M.C. AXTELL, J.A. STICKLES, *Zero-divisor graphs in commutative rings*, Commutative Algebra, Noetherian and Non-Noetherian Perspectives (M. Fontana, S.E. Kabbaj, B. Olberding, I. Swanson) Springer-Verlag, New York, 2011, 23-45.
- [3] D. ARCHDEACON, *A Kuratowski Theorem for the projective plane*, J. Graph Theory **5** (1981) 243-246.
- [4] I. BECK, *Coloring of commutative rings*, J. Algebra **116** (1998) 208-226.
- [5] J.A. BONDY, U.S.R. MURTY, *Graph Theory with Applications*, American Elsevier, New York, 1976.
- [6] A. BOUCHET, *Orientable and nonorientable genus of the complete bipartite graph*, J. Combin. Theory Ser. B **24** (1978) 24-33.
- [7] H.J. CHIANG-HSIEH, P.F. LEE, H.J. WANG, *The embedding of line graphs associated to the zero-divisor graphs of commutative rings*, Israel J. Math. **180** (2010) 193-222.
- [8] B.A. DAVEY, H.A. PRIESTLEY, *Introduction to Lattices and Order*, Cambridge University Press, 2002.
- [9] H. GLOVER, J.P. HUNEKE, C.S. WANG, *103 graphs that are irreducible for the projective plane*, J. Combin Theory Ser. B **27** (1979) 332-370.
- [10] C.D. GODSIL, G. ROYLE, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.
- [11] A.V. KELAREV, S.J. QUINN, *Directed graphs and combinatorial properties of semigroups*, J. Algebra **251** (2002) 16-26.

- [12] A.V. KELAREV, J. RYAN, J. YEARWOOD, *Cayley graphs as classifiers for data mining: the influence of asymmetries*, Discrete Math. **309** (2009) 5360-5369.
- [13] K. KHASHYARMANESH, M.R. KHORSANDI, *Projective total graphs of commutative rings*, Rocky Mountain J. Math. (to appear).
- [14] W. MASSEY, *Algebraic Topology: An Introduction*, Harcourt, Brace & World, New York, 1967.
- [15] W. MYRVOLD, J. ROTH, *Simpler projective plane embedding*, Ars Combin. **75** (2005) 135-155.
- [16] G. RINGEL, *Map Color Theorem*, Springer-Verlag, New York/Heidelberg, 1974.
- [17] J. SEDLÁČEK, *Some properties of interchange graphs*, Theory of graphs and its applications, (M. Fiedler, ed.) Progue 1962; Reprinted, Academic Press, New York (1962), 145-150.
- [18] Z. XUE, S. LIU, *Zero-divisor graphs of partially ordered sets*, Appl. Math. Lett. **23** (2010) 449-452.

**MATHEMATICAL MODELING OF THE SYSTEM OF
CHAINS OF COUPLED ELEMENTS WITH STRONG ANTICIPATION¹****A. Makarenko**

NTUU "KPI", Institute for of Applied System Analysis
Prospect Peremogy 37, 03056
Kiev-56, Ukraine
makalex@i.com.ua

Received February 10, 2015

Abstract

Since the introduction of strong anticipation by D.Dubois the numerous investigations of concrete systems had been proposed. But further development of the theory of anticipatory systems follows to the investigations of new examples of distributed in space systems with anticipation. So, in proposed paper the new examples of distributed physical models with anticipation had been considered - namely the system of chains of coupled elements with strong anticipation. It is proposed the mathematical formulation of problems, possible analytical formulas for solutions and examples of presumable distributed solutions including synchronization. Multivalued behavior of such solutions is discussed including analogs of 'chimera' states.

Keywords: Chains, synchronization, anticipation, multivalued solutions, complex solutions, nonhomogeneity, 'chimera' states.

¹Based on invited talks given by A. Makarenko at the 6th Petrov International Symposium on *High Energy Physics, Cosmology and Gravity*, Kyiv-Kosivska Poliana (Ukraine), September 5-20, 2013 and at INDS11 (Klagenfurt, Austria).

1 Introduction

Complexity and synchronization phenomena in technical and natural systems now became one of the key topics for investigations in philosophy, physics, humanity and biological sciences. The reason is in the globalisation of processes, hierarchical structures with many levels, interconnections between elements, existence of many sub-processes with different time and space scales.

The synchronization theory has about two centuries history and more or less well defined list of topics (see for examples ([4], [7]-[9], [21], [22], [35], [36], [38], [42]) and many others). Usually the objects of synchronization research are the collections of elements with internal dynamics with a set of bounds between elements and types of behavior in such systems.

Remark that usually the main sources of synchronization theories are natural, physical, technical systems and problems: mechanics, electrical and electronic circuits, biology, medicine, neuroscience etc. Following the line of synchronization theory development conditionally some more or less defined steps of development can be selected

I. Interpretation of real system behaviors and searching new features, ideas and models concepts.

II. Formalization and formulating of mathematical problems.

Introducing the formal description of elements states with new mathematical properties.

Mathematical description of single element behavior.

Posing the formal mathematical problems on the behavior of coupled elements systems.

III. Mathematical investigation of synchronization problems.

Investigation of single element behavior (theory and computational experiments).

Investigation of few coupled elements behavior.

Investigation of large systems of coupled elements.

Searching and formalizing of new presumable types of behavior.

Strict mathematical results of synchronization peculiarities.

IV. Interpretation of solutions of new class of systems, posing new research problems, including real system design and control approaches.

As the examples of steps of development of synchronization theory we may remark at first the synchronization of chains and lattices of mathematical pendulums ([22], [35], [36]) and synchronization in the chains of coupled maps ([21], [34], [35]). At first case the basic model of element is usual equations of mathematical pendulum. In second – the maps of interval.

Usually the objects of synchronization research are the collections of elements with internal dynamics and a set of bounds between elements. Presence of the bounds follows to the including into the equations special terms. Most common are two types of equations (with continuous and discrete time). The example of object of first type is

$$\frac{du_i}{dt} = f(u_i, \lambda) + \varepsilon g(\{\vec{u}\}, \theta), \quad i = 1, 2, \dots, N, \quad (1)$$

where u_i - the state of i -th element, $\{\vec{u}\}$ - the collection of states of elements, N is the number of elements in the system, function f corresponds to description of internal dynamics of elements; g corresponds to the bonds between elements; $\{\lambda, \varepsilon, \theta\}$ - the set of parameters; ε corresponds to the strength of interconnections.

The second common example for synchronization investigations is the chain of couple maps ([35], [36]):

$$u_i(n+1) = f(u_i(n), \alpha) + \varepsilon g(\{\vec{u}\}, \theta) \quad (2)$$

where $u_i(n) = u_i(\tau n)$, τ - time step. The most investigated is homogeneous case with the same coefficients for all elements. But recently also non-homogeneous cases are under investigation (including stochastic case).

The steps I - IV had been investigated by mane researches and had been described in many publications. But since the initial period of synchronization theory many new classes of synchronous objects aroused. Strict mathematical definitions of synchronization and its types had been proposed [35], [36]. Also the behavior of its solutions, dependence of parameters, control problems in synchronization (including chaotic systems) is investigated. During synchronization investigations also very important are numerical investigations of systems. Especially interesting is synchronization phenomena in spatially distributed systems (including differential equations,

inclusions and networked systems). The bright recent example of computations is 'chimera' states introduced by Y. Kuramoto with colleagues [1]. Other very prospective class of the synchronized systems is cellular neural networks introduced by L. Chua [9], [11].

Recent development of the theory and the practice in the field of synchronization has many different ways. Usually the nature of specific field of investigations follows to the specific new research problems and new objects. One of recent examples is the chain of chaotic Lorentz systems.

One special and large class of systems with synchronization is the systems with delay (or by other words the systems with the memory). The memory is very general property of all systems (natural, physical, technical) [13], [37], [45]. Many investigations in synchronization of delay systems had been proposed since the 1970th. Recently some very interesting papers are proposed especially by Chine's researchers in the field of synchronization of delayed and impulsive systems [9], [37], [42], [43].

But more and more new properties of the systems, elements, behavior of the solutions should be taken into account. One of the important examples are quantum systems.

But recently it has been found that also very interesting are the investigation of the systems with anticipation. The term 'anticipation' had been firstly attached to the systems with the intrinsic models for predicting the evolution of the systems and of their environment [2], [14]-[16], [26], [39] and many others. But now since the works by Daniel M. Dubois (Belgium) the notion of 'strong anticipation' had been introduced and investigated [14]-[16]. In case of strong anticipation the system haven't the models for predictions but are self-making with accounting presumable future states of the system. D. Dubois also introduced the notion of incursive and hyper-incursive systems (with presumable multivaluedness of solutions). Also D. Dubois firstly described the elementary single element with hiperincursion. One of the most important examples of the models with anticipation follows from modeling of large social systems. Some examples of investigations of the systems with anticipation had been described in [23], [26]-[32]. Remark that one of the most new and interesting properties in such systems is presumable multivaluednes of the solutions (that is existing of some values of the solution at each moment of time). Because of this it is very prospective

to consider the properties of multivalued solutions from the point of view of synchronization investigations.

So in the proposed paper the next issues will be represented: The description of strong anticipation following D.Dubois, general example of network of elements with strong anticipation; examples of particular synchronization problems in the system with few elements; examples of multivalued solutions in different systems with anticipation and some presumable types of solutions behavior; presumable possibilities of manifestation of such kind of synchronization and further research problems. Thus the given paper should be considered as the beginning of realizations of the steps I - IV in investigation synchronization in new class of systems namely the in the systems of coupled anticipated elements.

2 Strong anticipation property

Since the beginning of 1990-th in the works by D.Dubois – see [14]-[16] the idea of strong anticipation had been introduced: “Definition of an incursive discrete strong anticipatory system ...: an incursive discrete system is a system which computes its current state at time t , as a function of its states at past times, $\dots, t-3, t-2, t-1$, present time, t , and even its states at future times $t+1, t+2, t+3, \dots$

$$x(t+1) = A(\dots, x(t-2), x(t-1), x(t), x(t+1), x(t+2), \dots, p) \quad (3)$$

where the variable x at future times $t+1, t+2, t+3, \dots$ is computed in using the equation itself, A is the operator of the system, p – the set of parameters.

Definition of an incursive discrete weak anticipatory system: an incursive discrete system is a system which computes its current state at time t , as a function of its states at past times, $\dots, t-3, t-2, t-1$, present time, t , and even its predicted states at future times $t+1, t+2, t+3, \dots$

$$x(t+1) = A(\dots, x(t-2), x(t-1), x(t), x^*(t+1), x^*(t+2), \dots, p) \quad (4)$$

where the variable x^* at future times $t+1, t+2, t+3, \dots$ are computed in using the predictive model of the system” [15], (Dubois, 2001, p. 447).

Thus as the further research problem in the field of synchronization should be considered synchronization in the system with strong anticipation. The simplest but rather general counterparts for the equations (1) and (2) are the equations for continuous time case and for discrete time case accounting strong anticipation (3) with single anticipation time.

The example of object of those types (counterpart to equation (1)) is

$$\frac{du_i}{dt} = f(u_i, u_i(t + \tau), \lambda) + \varepsilon g(\{\vec{u}\}, \{\vec{u}(t + \tau)\}, \theta), \quad i = 1, 2, \dots, N \quad (5)$$

where N is number of elements in the system, functions f_i correspond to description of internal dynamics of elements; g corresponds to the bonds between elements; $\{\lambda, \varepsilon, \theta\}$ - the set of parameters, τ - time of anticipation.

The second common example for synchronization investigations is the chain of couple maps with strong anticipation on one forward discrete step:

$$u_i(n + 1) = f(u_i(n), u_i(n + 1), \alpha) + \varepsilon g(\{\vec{u}\}, \{\vec{u}(n + 1)\}, \theta) \quad (6)$$

where $u_i(n) = u_i(\tau n)$, γ -time step.

The equations (5), (6) are rather new mathematical objects and their full considerations are the tasks for further investigations. So here we had described only first results of such chains investigations. We describe the results of complex behavior research for discrete maps, differential equations, artificial neural networks, cellular automats. The complex branching non-homogeneous solutions also are proposed.

3 Examples of investigated problems with anticipatory property

In this section we briefly describe for illustration the possibilities of strong anticipation property manifestation in some problems.

3.1 Two-steps discrete-time anticipatory models

The proposed model has two features (see details at [33]). The first is two-steps nature (that is passing on two steps ahead at discrete). The second

is that the function $f(x)$ in the model has a piecewise-linear character, and looks like the transition function of neurons in neuronets [9]. Remark that the piecewise character of the nonlinearity usually allows to simplifying mathematical investigations (see for example [5], [9], [34], [35]).

We write down the proposed model as follows:

$$\begin{cases} p_{t+1} = \alpha \cdot m + (1 - \alpha)p_t - \alpha \cdot f(p_{t+2} - p_{t+1}) \\ p_{t+2} = \alpha \cdot m + (1 - \alpha)p_{t+1} - \alpha \cdot f(p_{t+2} - p_{t+1}) \end{cases} \quad (7)$$

The coefficient α corresponds to accounting of the anticipation. (The case $\alpha = 0$ corresponds to the absence of anticipation). The function $f(x)$ depends on the parameter α and has the following expression:

$$\begin{cases} f(x) = 0 & x \leq 0 \\ f(x) = \alpha \cdot x & x \in (0, \frac{1}{\alpha}] \\ f(x) = 1 & x > \frac{1}{\alpha} \end{cases}$$

The software had been developed so that it is possible to visualize some branches of the states of map, but only those, that are not thrown out to infinity. This facilitates the understanding of the processes, which take place in the region of ambiguousness. Namely, at the Figure 1 below we can see branching of solution at discrete time steps. In other calculations we had seen the number of different period of cycles and increasing of the number of cycles in the course of time, that is we had seen a tendency to phenomena which we may called "chaos".

3.2 Cellular automata with anticipation

Recently we have investigated some kind of such models. With anticipation accounting: game 'Life' with anticipation [31]; movement of pedestrian crowds [32]; general problems [23].

First of all let's make a short definition of cellular automata. The cellular automata are discrete dynamic system which represents set of identical cells connected among them. Cells create a lattice of the cellular automata. Lattices can be different types, differing both on the dimension, and under the form of cells [10], [23], [44].

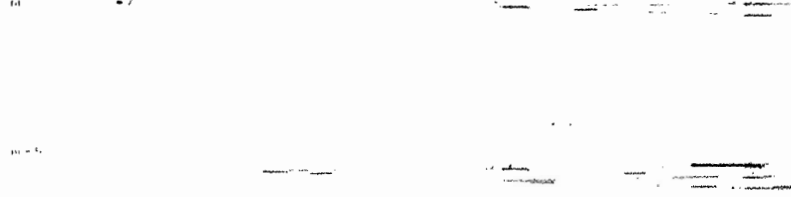


Figure 1: The graph of branching of solution. The parameters values are: $\alpha = 2$, $m = 0$. Limitations for visualization of the values of variables are the next: maximal $m + 7$ and minimal $m - 5$, without representing the branches, which are thrown out to infinity.

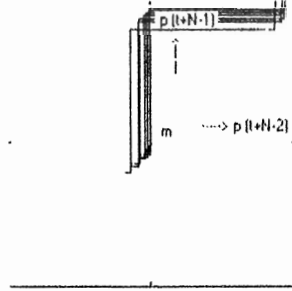


Figure 2: Map in state space. The sequence of values of (p_t, p_{t+1}) is represented for the solution from Fig. 1.

The local rule for cell k on the Z_d is the transformation T_k which transforms the state $s_k(t)$ in S of cell k at moment t to the state $s_k(t+1)$ in S of the same cell at moment $(t+1)$.

$$S_i(t+1) = G_i(\{s_i(t)\}, R), \quad (8)$$

where N_k – some neighborhood of cell k on the lattice Z_d ; $\{s_k(t)\}$ is the set of cell's states within N_k , the transformation T_k result depends only on the states of elements within the neighborhood N_k (locality), R – some parameters for rules which define transformations. The collection of local transformations T_k define the global transformation G on the configuration

space C

$$C(t+1) = G(C(t));$$

The initial data $C(0)$ configuration is defined at initial moment $t = 0$. The set of transformations $\{T_k\}$ or transformation G defines the cellular automata on the lattice Z_d with the cell's state space S .

3.2.1. Game 'Life' as example of CA

One of the basic examples of CA is well-known Game 'Life' by J.Conway [10], [17], [18], [44]). At a time t let some subset of the cells in the array are living. The living cells at time $t + 1$ are determined by those at time t according to the following evolutionary rules (the details are posed at next section of paper). Remark that anticipation follows to the origin of equations of type:

$$s_i(t+1) = G_i(\{s_i(t)\}, \dots, \{s_i(t+g(i))\}, R). \quad (9)$$

In equation (9) $\{s_k(t)\}$ is the collection of element states at discrete moment t , G_i is the function of i -th element, R – the set of parameters. Remark (see [23], [25]-[31]) that the main peculiarities of such equation is that such equations may have multivalued solutions (of course in some range of parameters).

Graph in Figure 3 presents a case of well-developed solution multivaluedness within the model "LifeA".

Such objects are very interesting and prospective for investigations of synchronizations. Remark that the main peculiarities in such investigated systems also are presumable multivaluedness and space non-homogeneity of solutions behavior.

3.3 Neural networks with anticipation

Other very important object of proposed type (with anticipation and discrete time dynamic) is neural networks with anticipation. We already have investigated some such models [30]. Here we propose for understanding of presumable behavior only very short description of calculations. One of the

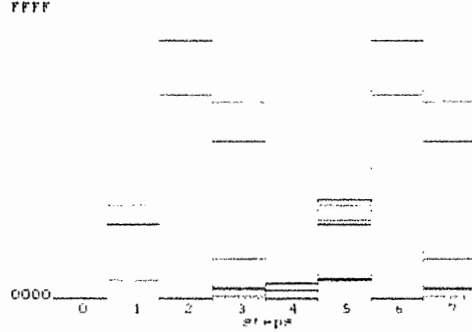


Figure 3: A large number of configurations coexisting in system (additive next state function, $\alpha = 0.5$, initial state is 0000). The horizontal axe corresponds to the discrete time. The symbols at vertical axe (0000, ... , FFFF) correspond to the indexing of the configuration on 16^{th} cells space (with periodic boundary conditions) by 4-symbols words in alphabet.

simplest variant of such models with anticipation accounting has the form (counterpart for Hopfield's [20], [30] neural networks)

$$x_j(n+1) = f \left((1-\alpha) \sum_{i=1}^N w_{ji} x_i(n) + \alpha \sum_{i=1}^N w_{ji} x_i(n+1) \right), \quad (10)$$

where α is the anticipation parameter. The case of $\alpha = 0$ corresponds to the absence of anticipation (with classical neuronet). $x_i(n)$ is the state of i -th element at time moment n ; f is the transition function of element; w_{ji} , $i = 1, 2, \dots, N$; $j = 1, 2, \dots, N$ is the value of bonds between i -th and j -th elements. $W = \{w_{ji}\}$, $i = 1, 2, \dots, N$; $j = 1, 2, \dots, N$ is the connections matrix of neuronet. Sigmoidal and piece-wise linear sigmoidal function had been used in the numerical calculations.

Below at the picture (Fig. 4) we propose the example of presumable behavior of the model (10) at different time moments.

The non-homogeneous in space solution is seen. Two properties of artificial neural network solutions are represented at the Fig. 4: multivaluedness and non-homogeneities of elements behavior. Both are rather new and

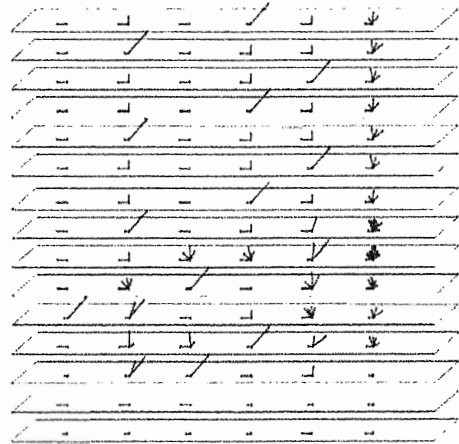


Figure 4: Behavior of multivalued solution. The numbers at right side of figure corresponds to the discrete time moments. The network had 6 elements. The number of arrows adjusted to each of 6 points corresponds to the number of solution branches (values) of each of the elements. The length of the arrow displays the value of the state.

prospective for development of synchronization investigations as in theory as in applications.

3.4 Presumable properties of solutions and further research problems

Here we pose some comments which may be useful also for considering synchronization problem. The detailed consideration of such multivalued dynamical systems may constitute the subject of further mathematical investigations. Here we only pose some general comments.

Because the map acts on the multivalued functions of $[0, 1]$ than the dynamical system is infinite-dimensional. Very important is the asymptotic solutions when $t \rightarrow \infty$. For single-valued case (without anticipation) by A. Sharkovsky and other [40] it has been found the existence of so called relaxation oscillation regimes when the solution tends to the function with

finite number of discontinuities at fix length time interval and so-called turbulent solutions when the number of discontinuities tends on fix length interval tends to infinity as $t \rightarrow \infty$. Some limit objects for such solution may have fractal structure. Moreover it has been proved that the limiting regime is multivalued function for which some probability distribution corresponds. And finally very interesting for such problems is the structure of attractors in single-valued case. It had been found that attractors are infinite-dimensional and contained generalized functions (A. Sharkovsky, Yu. Maistrenko, E. Romanenko). The attractors are with simple dynamics: "... the movement on attractors is rather simple – periodical or almost periodical, also the elements of attractor are functions acting from $I = [0, 1]$ into I , may be very complex (for example with cantor set of multivaluedness points" [40].

Based on our results it may be proposed some generalizations for cases with accounting of anticipating. At first as we described above the solutions in such case may be multivalued. So the fractal dimension of solutions may be bigger then that in single-valued case. Moreover the behavior of the solutions may be much more complex. Different regimes on the different branches of solutions may exist (some examples of such different branches see in [33]). So each branch may have different stochastic properties, different types of relaxation behavior etc. But may be the most interesting is the common behavior of mixture of the branches in multivalued case. Especially interesting is the problem of limiting behavior as $t \rightarrow \infty$: limit solutions, attractors of such solutions, probability properties of limiting objects.

But also the new problems arrows: observability of solutions, selection the single-valued trajectories of the system, its limiting objects complexity and such complexity measures, searching adequate mathematical spaces for considering the problems and solutions. Just the problems of adequate definition of periodic behavior (and moreover of dynamical chaos), Lemeray's staircase, Poincare's bifurcation diagram, ergodicity, mixing are interesting.

4 Examples of synchronization problems formulations with anticipatory property

Based on presumable new peculiarities of the solutions from Section 3 here we display some examples of synchronization problems from general classes (5) and (6) which are especially interesting and typical for their investigations. Remark that in this paper we pose only the short outline of the presumable form of research problems and sometimes short comments.

We should stress that the equations with anticipations have attracted before less attention then the equations with delay as in mathematics as in physics and technique. So a lot of before already solved problems in the theory of equations with delay require new re-investigations for the equations with anticipation. Moreover frequently just the posing strict mathematical problems require introducing of new definitions and formalization of known notions (as the examples we remember the chaos in our case and just the notion of periodic regimes).

So, below we describe briefly some presumable research problems and mathematical objects for investigations.

4.1 One discrete element with strong anticipation

As the first important step in the investigations of the system with anticipation we should propose the mathematical investigation of single discrete time equation:

$$u(n+1) = f(u(n), u(n+1), \alpha) \quad (11)$$

First very important examples of mathematical objects of such kinds are the counterparts of well known logistic (Ferhulst equation) [21], [22]:

$$x(n+1) = \lambda x(n)(1 - x(n)),$$

where λ is the parameter. The counterparts of logistic equation with anticipation (11) may have a large number of forms, but some of the most natural are the next (α - the coefficient of anticipation):

$$\begin{aligned} x(n+1) &= (1 - \alpha)\lambda x(n)(1 - x(n)) + \alpha\lambda x(n+1)(1 - x(n+1)) \\ x(n+1) &= \lambda x(n)(1 - x(n)) + \alpha\lambda x(n+1)(1 - x(n+1)) \end{aligned}$$

Remark that in [25] some investigations of counterpart with anticipation of well-known classical logistic equation [40], [41]

$$x(n+1) = \lambda \cdot x(n) \cdot (1 - x(n)) - \alpha x^2(n+1))$$

had been investigated.

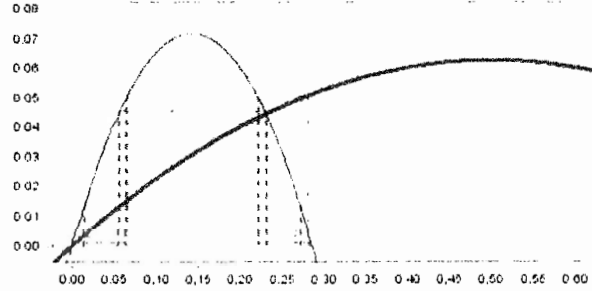


Figure 5: Behavior of multivalued solution of logistic equation with anticipation ($\lambda = 0.25$, $\alpha = -3.5$). The picture displays the first 5 iterations (analog of Lemerey's staircase for multivalued case accounting existing of different branches of the solution).

Counterparts of periodic and chaotic behavior in multivalued case had been investigated numerically and analytically.

4.2 Two coupled elements with strong anticipation

The next object for synchronization investigations is the coupled system of two identical elements described by equation [11]:

$$u_1(n+1) = f(u_1(n), u_1(n+1), \alpha) + \varepsilon g(u_2(n), u_2(n+1), \theta) \quad (12)$$

$$u_2(n+1) = f(u_2(n), u_2(n+1), \alpha) + \varepsilon g(u_1(n), u_1(n+1), \theta) \quad (13)$$

In the formulas (12), (13) the constant ε and function g correspond to the accounting of coupling between elements. Remark that all existing in classical case methods of investigations should be expand to the case of equations (12), (13) to the multi-valued cases: Lyapunov numbers, renorm-group methods, variational methods, ergodic theory, symbolic dynamics, complexity etc.

4.3 One ordinary differential equation with anticipation

The next important object is the ordinary differential equation with simplest anticipation accounting (also named advanced ordinary differential equation):

$$\frac{du}{dt} = f(u, u(t + \tau), \lambda) \quad (14)$$

We should remark that relatively small number of investigations had been devoted for such objects. The most of the investigations had been concerned to the problems of existence of the solutions (see [12], [24]). Also some existence theorems had been proposed for abstract operator equations of the form similar to (14). A little is known about the qualitative behavior of solutions of equation (14). Investigations of definitions and properties of analogs of deterministic chaos, phase portraits, bifurcations etc in multivalued are difficult problem. But in principle the techniques which had been developed for ordinary differential equations with discontinuous right hand [5] or differential inclusions approach may be used [6]. In such case, the numerical experiments could be proposed as the useful tool for investigations. But of course the building of adequate numerical methods just for equation (14) is new and difficult task.

4.4 Two coupled ordinary differential equations with anticipation

The next object for investigations is the system of two coupled elements with the equations (14):

$$\frac{du_1}{dt} = f(u_1, u_1(t + \tau), \lambda) + \varepsilon g(u_2, u_2(t + \tau), \theta) \quad (15)$$

$$\frac{du_2}{dt} = f(u_2, u_2(t + \tau), \lambda) + \varepsilon g(u_1, u_1(t + \tau), \theta) \quad (16)$$

In such case all existing in classical cases (without anticipation or with the delay property) problems can be proposed for investigations. As specific new research problem we can propose the investigations of the analogs

of single-valued anticipatory control (with master and slave equations) in multi-valued case.

4.5 Chain of coupled piece-wise linear elements

Fortunately much more investigated and narrower class of coupled elements exists - namely Chua circuits and CNN (cellular neural networks) [9], [11]. Frequently coupled ordinary differential equations with piece-wise nonlinearities are considered in such context. So the piece-wise linear functions are useful for considering dynamical behavior in space and time by considering the linear equations with prescribed coefficients in different space regions. Any existing models of such kind may be transformed to the models with anticipation by introducing anticipation to the right hand of equations. For example, the simplest example of two coupled elements follows easy to the system with anticipation.

4.6 Coupled Chains of general maps with anticipation

The counterpart of the chain (6) without anticipation in general case has the form:

$$u_i(n+1) = f_i(u_i(n), u_i(n+1), \alpha_i) + \vec{\varepsilon}_i g_i(\{\vec{u}\}, \{\vec{u}(n+1)\}, \theta_i), \quad (17)$$

where the parameters of elements, functions, coupling constants may be different for different elements (including random and 'small world' structures of the systems [4]).

4.7 Chains of ordinary differential equations

Next evident example is the chain of coupled elements which have the internal dynamics described by ordinary differential equation. As the first counterpart with anticipation we may propose to take the models of synchronization with delay in elements (see for example [10], [37], [43]) but with the delays replacing by anticipation in equations for dynamics of elements.

But because of such objects novelty just two coupled ordinary differential equations with anticipation are also very interesting. For example just two

coupled oscillators with anticipation constitute interesting and important research problem. The first steps of such investigations are described for example at [3].

4.8 General distributed media with anticipation

Much more models may be proposed for the case of considering the continuous media with some kind of anticipation. The simplest variant is when only the nonlinear source $f(u)$ in equations of such media (where $u(x, t)$ is field value) has anticipatory property, that is $f_1 = f(u(x, t), u(x, t + \tau))$

in simplest case or $f_2 = \int_{-\infty}^{+\infty} f(u(x, \tau))K(x, t - \tau)d\tau$ in more complex case.

The example of such kind of model is parabolic equation of diffusion with such nonlinear source. As the example of such system we may propose the models with continuous variables of neural activation fields (counterparts for known models of S. Amari or H. Wilson & J. Cowan). Remark that much more complex and developed models of such kind phenomena are the counterpart to well known [13], [45] strict models to the media with non-locality and memory in theoretical physics.

4.9 ‘Chimera’ states in the systems with anticipation

Another very prospect issues for investigations in the fields of synchronization are the analogs of ‘chimera’ states in case of anticipative systems. Remember that ‘chimera’ states are highly non-homogeneous transitive solutions when different types of behavior coexist in space [46], [47].

The ‘chimera’ states are the solutions in chains of elements or in the distributed media which have the different behavior in different domains of the space. For example such systems can have coexisting domains with chaotic and ‘smooth’ behavior in different places of the spaces. One of the discrete systems with presumable ‘chimeras’ is the next [46]:

$$z_i^{t+1} = f(z_i^t) + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} (f(z_j^t) - f(z_i^t)) \quad (18)$$

The next example is the chains of oscillators:

$$\frac{d\psi_k(t)}{dt} = \omega - \frac{1}{2R} \sum_{j=k-R}^{k+R} \sin(\psi_k(t) - \psi_j(t) + \alpha) \quad (19)$$

The next general example with 'chimera' state are the next [48], [49]:

$$\begin{aligned} \partial_t X(r, t) &= f(X, Y) + K(B - X) \\ \partial_t Y(r, t) &= g(X, Y) \\ \tau \partial_t B(r, t) &= -B + D \nabla^2 B + X \end{aligned} \quad (20)$$

and (for example) more developed [50]:

$$\begin{aligned} \partial_t \eta &= (1 + i\omega)\eta - (1 + i\alpha)|\eta|^2\eta + (1 + i\beta)\nabla^2\eta + \\ &+ K \int_{-\infty}^{+\infty} \int_0^t G(x - x', t - t')(\eta(x, t) - \eta(x', t')) dx' dt' \end{aligned} \quad (21)$$

It is accepted that the main source of existing of 'chimera' states is the nonlocality in equations (see as the examples (18), (19)). Remark that the nonlocality described in the [13] essentially expand the cases with presumable origing of 'chimeras'. But absolutely new are the possibilities of 'chimeras' in the systems with the anticipation. As the examples we should remark the possibilities of 'multivalued chimeras', coexisting of different 'chimeras' on different branches of the multivalued solutions, coexisting of 'chimeras' and 'smooth' behavior in the different branches of the solution. As one of the presumable basic system with anticipation for investigation of 'chimeras' we can propose the next:

$$\begin{aligned} z_i^{t+1} &= (1 - \alpha) \left(f(z_i^t) + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} (f(z_j^t) - f(z_i^t)) \right) + \\ &+ \alpha \left(f(z_i^{t+1}) + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} (f(z_j^{t+1}) - f(z_i^{t+1})) \right) \end{aligned} \quad (22)$$

The next example is the chains of anticipatory oscillators:

$$\begin{aligned} \frac{d\psi_k(t)}{dt} = & \omega - \frac{1-\alpha}{2R} \sum_{j=k-R}^{k+R} \sin(\psi_k(t) - \psi_j(t) + \alpha) + \\ & + \frac{\alpha}{2R} \sum_{j=k-R}^{k+R} \sin(\psi_k(t+\tau) - \psi_j(t+\tau) + \alpha) \end{aligned} \quad (23)$$

The next general simplest example with 'chimera' state are the next:

$$\begin{aligned} \partial_t X(r, t) &= f(X, Y) + K(B - X) + \alpha K(B(r, t + \gamma) - X(r, t + \gamma)), \\ \partial_t Y(r, t) &= g(X, Y) + \alpha g(X(r, t + \mu), Y(r, t + \mu)), \\ \tau \partial_t B(r, t) &= -B + D \nabla^2 B + X + \alpha X(r, t + \delta), \end{aligned} \quad (24)$$

where δ, γ, u some positive constants. Other example is

$$\begin{aligned} \partial_t \eta &= (1 + i\omega)\eta - (1 + i\alpha)|\eta|^2\eta + (1 + i\beta)\nabla^2\eta + \\ &+ K \int_{-\infty}^{+\infty} \int_0^t G(x - x, t - t)(\eta(x, t) - \eta(x, t))dxdt + \\ &+ K \int_{-\infty}^{+\infty} \int_t^\alpha G(x - x, t - t)(\eta(x, t) - \eta(x, t))dxdt \end{aligned} \quad (25)$$

where $\alpha > t$, α may be finite positive or just infinite. Of course, the models may be much more complex. The main presumable peculiarities may be the multivaluedness of 'chimeras'. The interesting research problems are the averaging on the branches, measurements of states, choosing the relevant physical selector of the solution.

4.10 Algebraic and geometrical aspects of the synchronization in the systems with anticipation

It should be remarked that the classical synchronization problems include explicitly or implicitly the algebraical and geometrical aspects. On of the

bright examples are the Lie symmetries of ordinary differential equations. Accounting of strong anticipation problems follows to new aspects of classical problems and to the posing of new research problems. Here we briefly remark some of them.

1. In case of the equations with strong anticipation presumable multivaluedness follows to the needs of investigation of generalization of semigroups defined by equations. In the case of multivaluedness it is necessary to consider the multivalued semi-flows [51], [52]. This can help to understand the properties of corresponding attractors.

2. In Lie theory of differential equations the last are represented by special geometrical surfaces in extended space which include the higher derivatives of the solutions. In case of anticipatory systems the theory requires the extension of such constructions on the multi-surfaces (see for example [53]).

3. The concepts of information and complexity also require generalization on the case of multivalued objects and processes. Especially prospect is the approach from [54] where the non-invariance of the objects (under chosen symmetries) took as the basic properties related to the complexity. The symmetries in single-value case had been related with the acting of corresponding algebraic objects (mostly with group actions). For multivalued case the generalization of groups is required. As the one of the possibilities we can remark the transformation of branching structures [55].

Conclusion

Thus in given paper we propose to consider the problem of complex behavior and synchronization for new class of objects namely for the chains and networks with strong anticipation. The presumable multivaluedness of solutions follows to new interesting properties in the frame of already existing concepts. But also new features can appear (for example non-heterogeneous multivalued synchronization) which are very prospect for further investigation. Also it is evidence the needs in applications of multi-valued analogs of groups, geometries and symmetries of multi-valued objects, including multivalued flows and semi-flows. Also the new may be considering the blow-up

solutions in multi-valued cases, including blow-up of selected branch of the solution.

We described only the first results of investigations and only some new presumable forms of research problems. But evident mathematical novelty of proposed problems and presumable great importance of applications (for examples for social systems; computation and signal processing theory; consciousness investigations etc.) follows to needs of further development of investigations.

Acknowledgements: The author is very grateful for D. Krushinski, B. Goldengotin, V. Biliuga, N. Yatsuk, S. Lazarenko for collaboration. Also the author gratitude Yu. Maistrenko for the detailed description of the investigations in the 'chimera' states field and supplying copies of relevant papers.

References

- [1] Abrams D., Strogatz S. Chimera states for coupled oscillators. Phys. Rev. Letters, 2004. vol. 93. pp. 174102 - 174105.
- [2] Anticipator behavior in adaptive learning systems: from brains to individual and social behavior. M. Butz, Sigaud O., Pezzulo G., Baldassare G. (Eds.), Springer-Verlag, Berlin-Heidelberg, 2007.
- [3] Antippa A., Dubois D., Discrete Harmonic Oscillator: A Short Compendium of Formalism. CP 1303, Proceedings of CASYS'09 Int. Conf. ed. D.M.Dubois, AIP, vol. 1303, 2010. pp. 111-120.
- [4] Arenas A., Dfas-Guilera A., Kurths J., Moreno Y., Zhou C., Synchronization in Coupled Networks, Physics reports, 469, 2008. pp. 93-153.
- [5] Awrejcewicz J., Lamarque C.-H. Bifurcation and chaos in non-smooth mechanical systems. World Sci.: Singapore.
- [6] Blagodatskih V., Filippov A. Differential inclusions and optimal control. Proceed of Mathematical Institute of USSR Academy of Science, Moscow: 1985. vol. 169. pp. 194-252. [In Russian].

- [7] Boccaletti S., Kurths J., Osipov G., Valladares D.L. Zhou C.S. The synchronization of chaotic systems. *Physics Reports*, 366, 2002. pp. 1-101.
- [8] Calvo O., Chialo D.R., Eguilaz V.M., Missaro C., Toral R. Anticipated synchronization: A metaphorical linear view. *Chaos*, 14, n.1., 2004. pp.1-13.
- [9] Chua L. *CNN: a paradigm of complexity*, World Science, 1998.
- [10] Chua L., Sook Yoon, R. Dogaru A nonlinear dynamics perspective of Wolfram's new kind of science. Part I: Threshold of complexity. *Int. J. Bifurcation and Chaos*, 2002, vol. 12, pp. 2655 - 2766.
- [11] Chua L., Yang L., *Cellular Neural networks*. *IEEE Transactions Circuits, Systems*, vol.35, pp. 1257-1272, 1988.
- [12] Corduneanu C. *Functional equations with causal operations: theory and applications*. Taylor&Francis. 2001.
- [13] Danilenko V., Danevich T., Makarenko A., Skurativskyi S. Vladimirov V. *Self-organization in nonlocal non-equilibrium media*. Kyiv: S.I. Subbotin Institute of Geophysics, NAS of Ukraine, 2011. 333 p.
- [14] Dubois D., Generation of fractals from incursive automata, digital diffusion and wave equation systems. *BioSystems*, vol. 43, 1997. pp. 97-114.
- [15] Dubois D., Incursive and hyperincursive systems, fractal machine and anticipatory ence. Published by the American Institute of Physics, AIP Conference Proceedings 573, 2001. pp. 437 - 451.
- [16] Dubois D., Introduction to Computing Anticipatory Systems. *International Journal of Computing Anticipatory Systems*, 2, 3-14. (1998).
- [17] Game'Life' http://en.wikipedia.org/wiki/Conway%27s_Game_of_Life
- [18] Gardner M. The fantastic combinatorika of John Conway's new solitaire game 'Life'. *Scientific American*, 1970, vol. 223, n.4, pp.120-123.

- [19] Goldengorin B., Makarenko A., Smilianec N. Some applications and prospects of cellular automata in traffic problems Cellular Automata, Proceed. Int. Conf. ACRI'06, LNCS 4173, Springer, 2006. pp.532-537.
- [20] Haykin S. Neural networks: a comprehensive foundation. Prentice Hall, 1994.
- [21] Kaneko K. Coupled map lattices, spatial analysis. Wiley, 1993.
- [22] Kapitaniak T. Controlling chaos: theoretical and practical methods in non-linear dynamics. Academic Press, 1996.
- [23] Krushinskiy D., Makarenko A., Cellular Automata with Anticipation. Examples and Presumable Applications. AIP Conf. Proc., vol.1303, ed. D.M.Dubois, USA, 2010. pp. 246-254.
- [24] Lakshmikantham V., Leela S., Zahia Drici, Mcrae F. Theory of causal differential equations. Atlantis Publisher, USA, Austin. 2010.
- [25] Lazarenko S., Makarenko A., Investigation of Complex Multivalued Solutions in Discrete Dynamical Systems with Anticipation. Abstracts Book of 10th Int. Conf. CASYS 2011, Liege, Belgium, August 2011. 2011. 1p.
- [26] Makarenko A., Anticipating in modeling of large social systems - neuronets with internal structure and multivaluedness. International Journal of Computing Anticipatory Systems, 13, 77 - 92. (2002).
- [27] Makarenko A., Anticipatory agents, scenarios approach in decision-making and some quantum - mechanical analogies. International Journal of Computing Anticipatory Systems, 15, 217 - 225. (2004).
- [28] Makarenko A., Anticipating in modeling of large social systems - neuronets with internal structure and multivaluedness. International Journal of Computing Anticipatory Systems, 13, 77 - 92. (2002).
- [29] Makarenko A., Anticipatory agents, scenarios approach in decision-making and some quantum - mechanical analogies. International Journal of Computing Anticipatory Systems, 15, 217 - 225. (2004).

- [30] Makarenko A. Neural networks with anticipation and some problems of complexity theory. Proceedings of Int. Conf. on Complex Systems: Synergy of Control, Communications and Computing - COSY 2011, Ohrid, Macedonia, 2011. pp. 257 - 262.
- [31] Makarenko A., Goldengorin B., Krushinsky D., Game 'Life' with Anticipatory Property. Proceed. Int. Conf. ACRI'08, Eds. Umeo, H. et al, LNCS 5191, 2008. pp. 77-82.
- [32] Makarenko A., Krushinski D., Goldengorin B., Anticipation and Delocalization in Cellular Models of Pedestrian Traffic. Proceed. Of INDS 08, Klagenfurt, Austria, July 2008. Shaykewr-verlag, Aachen, 2008. p.61-64.
- [33] Makarenko A., Stashenko A. Some two- steps discrete-time anticipatory models with 'boiling' multivaluedness. AIP Conference Proceedings, vol.839, ed. Daniel M. Dubois, USA, 2006. pp.265-272.
- [34] Maystrenko Yu., Kapitaniak T. Different types of chaos synchronization in two coupled piecewise linear maps. Phys. Rev. E, 2002. vol. 54. pp. 3285 - 3298.
- [35] Mosekilde E., Maistrenko Yu., Postnov D., Chaotic synchronization: applications to living systems. World Science, Singapoure, 2002.
- [36] Pikovsky A., Rosenblum M., Kurths J. Synchronization: a universal concept in nonlinear science. Cambridge University Press, 2001.
- [37] Ponce M., Masoller C., Marti A.C. Synchronizability of chaotic logistic maps in delayed complex networks. Eur. Phys. Journal. Ser. B. B67, 2009. pp. 83-93.
- [38] Recent Advances in Nonlinear Dynamics and Synchronization: Theory and Applications. Eds. Kyamakyia K., Halang W.A., Unger H., Chedjou J.C., Rulkov N.F., Li Z., Springer, Berlin/Heidelberg, 2009. 404 p.
- [39] Rosen R., Anticipatory Systems. Pergamon Press. 1985.

- [40] Scharkovski A., Maystrenko Yu., Romanenko E. Discrete equations and their applications. Kluwer. 1993.
- [41] Schuster H. Deterministic dynamics. Wiley, 1988.
- [42] Synchronization: Theory and Application. Eds. Arkady Pikovsky, Yuri Maystrenko, NATO Science series II Mathematics, Physics and Chemistry, vol. 109, Kluwer AP. 2003. 258 p.
- [43] Weiwei Wang, Jindle Cao, Synchronization in an array of linearly coupled networks with time-varying delay. Physica A, 366, 2006. pp. 197-211.
- [44] Wolfram S. New kind of science. Wolfram Media Inc., USA, 2002. 1197 pp. (also at www.wolframscience.cm).
- [45] Zubarev D., Morozov V., Ropke G. Statistical mechanics of nonequilibrium process: basic concepts, kinetic theory. John Wiley&Sons. 1996.
- [46] I. Omelchenko, Yu. Maystrenko, P. Hovel, E. Scholl Loss of coherence in dynamical networks: Spatial chaos and chimera states Phys. Rev.Lett.,106: 234102, 2011.
- [47] M.J. Pannaggio, D. M. Abrams Chimera states: coexistence of coherence and incoherence in networks of coupled oscillators. arXiv: 1403.6204v2 [nlin.CD]12 May 2014. 18 p.
- [48] Kuramoto Y., Shin-ichiro Shima Rotating spirals without phase singularity in reaction-diffusion systems. Progress of Theoretical Physics Supplement. N. 150. 2003. pp. 115-125.
- [49] Shin-ichiro Shima, Kuramoto Yashiki. Rotating spiral waves with phase-randomized core in nonlocally coupled oscillator. Physica Review E., 2004. vol. E69. pp. 036213-1 - 9.
- [50] Casagrande V., Mikhailov A.S. Bhythmicity, synchronization, and turbulence in an oscillatory system with nonlocal inertial coupling. Physica D., 2005. vol. D205. pp. 154-169.

- [51] Melnik V.S., Valero J. On attractors of multivalued semi-flows and differential inclusions. Set-Valued Analysis, 1998. vol. 6. pp. 83-111.
- [52] Kapustyan O.V., Melnik V.S., Valero J., Yasinsky V.V. Global attractors of multi-valued dynamical systems and evolution equations without uniqueness. Kyiv, Naukova Dumka, 2008. 215 p.
- [53] Vinogradov A.M., Krasilshchik I.S., Lychagin V.V. Geometry of jet spaces and nonlinear partial differential equations. Gordon and Breach: N.Y. 1986.
- [54] Makarenko A. Complexity of individual objects without including probability concept, Asymmetry, observer and perception problem. Conf. Proceed. AIP, 2001. vol.573 . pp. 201-215.
- [55] Grigorchuk P.I. Branch group. Mathematical Notes, 2000, vol.67, N. 6. pp. 718-723.

ALGEBRAS GROUPS AND GEOMETRIES

INDEX VOL. 32, 2015

QUALITATIVE ANALYSIS OF MATHEMATICAL MODELS OF RELAXING MEDIA WITH SPATIAL AND TEMPORAL NONLOCALITY, 1

**Danylenko V.A., Danevich T.B., Makarenko O.S.,
Skurativsky S.I. and Vladimirov V.A.**

Subbotin Institute of Geophysics of the National

Academy of Sciences of Ukraine, Palladin Av. 32, Kyiv, Ukraine, UA-03680

MATHEMATICAL MODELLING IN HYDRODYNAMICS WITH MEMORY, 53

**Danylenko V.A., Danevich T.B., Makarenko O.S.,
Skurativsky S.I. and Vladimirov V.A.**

Subbotin Institute of Geophysics of the National

Academy of Sciences of Ukraine, Palladin Av. 32, Kyiv, Ukraine, UA-03680

MATHEMATICAL FOUNDATIONS OF SANTILLI ISOTOPIES, 135

Raúl M. Falcón Ganfornina*, Juan Núñez Valdés†

Alan Aversa (translator)‡

*Department of Applied Mathematics

E.T.S. de Ingeniería de Edificación

Universidad de Sevilla

Avda Reina Mercedes 4 A, 41012 Seville, Spain

rafalgan@us.es

†Department of Geometry and Topology

Faculty of Mathematics, Universidad de Sevilla

Apartado 1160, 41080 Seville, Spain

Department of Pure Mathematics, International Campus of Ferdowsi

Univeristy of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

MATHEMATICAL STRUCTURES IN COMPLEX BEHAVIOUR OF STRONGLY NONEQUILIBRIUM MEDIA, 309

**Christenyuk V.O., Danylenko V.A., Danevich T.B., Makarenko O.S.,
Skurativsky S.I. and Vladimirov V.A.**

Subbotin Institute of Geophysics of the National

Academy of Sciences of Ukraine, Palladin Av. 32, Kyiv, Ukraine, UA-03680

WEIGHT DISTRIBUTION OF LPA CODES AND THEIR DUALS, 369

Sapna Jain

Department of Mathematics, University of Delhi, Delhi, 110 007, India

Akshaya Kumar Mishra

Department of Mathematics, Berhampur University, Berhampur, Odisha, India

Kar-Ping Shum

Institute of Mathematics, Yunnan University, Kunming, 650091, P.R. China

**ADDITIVITY OF MAPS PRESERVING A SUM OF TRIPLE PRODUCTS
ON STANDARD OPERATOR ALGEBRAS, 387**

João Carlos da Motta Ferreira and Maria das Graças Bruno Marietto
Center of Mathematics, Computation and Cognition, Federal University of
ABC, Avenida dos Estados, 5001, 09210-580, Santo André, Brazil

**SCOTT CLOSED RETRACTNESS AND DIRECT SUMS OF DIRECTED
COMPLETE POSET ACTS, 403**

Mojgan Mahmoudi and Mahdiah Yavari
Department of Mathematics
Shahid Beheshti University, G.C., Tehran 19839, Iran

**LINE GRAPHS ASSOCIATED TO COMAXIMAL GRAPH OF LATTICES
OF GENUS ONE, 417**

K.H. Ahmad Javaheri
Department of Pure Mathematics, International Campus of Ferdowsi
University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

MATRIX RINGS OVER GENERALIZED (σ, δ) -RIGID RINGS, 437

V.K. Bhat and Meeru Abrol
School of Mathematics, SMVD University
Katra, J and K, India-182320

**MULTISCALE AND APPLIED SYSTEMS ANALYSIS OF A
BRAIN ACTIVITY. I. GENERAL DESCRIPTION, 449**

A. Makarenko
NTUU "KPI", Institute for of Applied System Analysis
Prospect Peremogy 37, 03056, Kiev-56, Ukraine

**RETRACTNESS OF COMPACT REVERSIBLE DIRECTED COMPLETE
POSET ACTS, 461**

M. Mehdi Ebrahimi and Mahdiah Yavari
Department of Mathematics, Shahid Beheshti University, G.C.
Tehran 19839, Iran

**ON THE PLANAR AND OUTER PLANAR ANNIHILATING-IDEAL
GRAPHS OF A LATTICE, 479**

F. Shahsavar
Department of Pure Mathematics, International Campus of Ferdowsi
University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

**MATHEMATICAL MODELING OF SPATIAL AND TEMPORAL
NONLOCALITIES ON NONLINEAR WAVES EVOLUTION, 495**

**Danylenko V.A., Danevich T.B., Makarenko O.S.,
Skurativsky S.I. and Vladimirov V.A.**
Subbotin Institute of Geophysics of the National Academy of Sciences of
Ukraine, Palladin Av. 32, Kyiv, Ukraine, UA-03680

**PLANAR AND PROJECTIVE LINE GRAPHS ASSOCIATED
TO ZERO DIVISOR GRAPHS OF PARTIALLY ORDERED SETS, 525**

A. Parsapour

Department of Pure Mathematics, International Campus of Ferdowsi
University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

**MATHEMATICAL MODELING OF THE SYSTEM OF CHAINS OF
COUPLED ELEMENTS WITH STRONG ANTICIPATION, 545**

A. Makarenko

NTUU "KPI", Institute for of Applied System Analysis
Prospect Peremogy 37, 03056, Kiev-56, Ukraine